REAL TIME DIGITAL SIGNAL PROCESSING

UTN-FRBA
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Adaptive Filters

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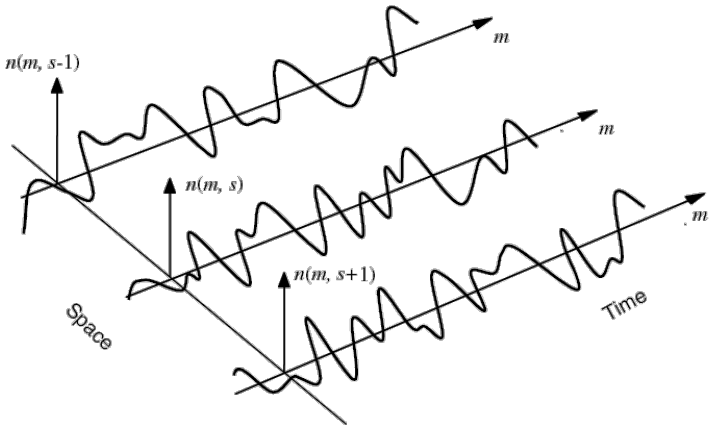
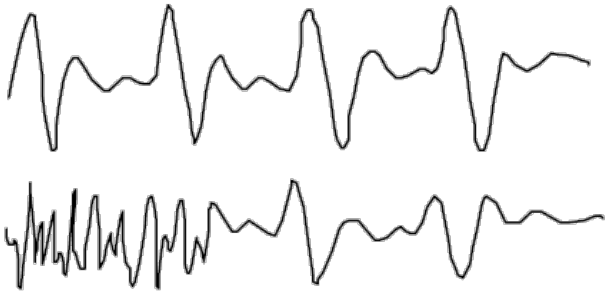
Stochastic Processes

- The term “stochastic process” is broadly used to describe a random process that generates sequential signals such as speech or noise.
- In signal processing terminology, a stochastic process is a probability model of a class of random signals, e.g. Gaussian process, Markov process, Poisson process,etc.



Stationary and Non-Stationary Random Processes

- The amplitude of a signal x(m) fluctuates with time m, the characteristics of the process that generates the signal may be time-invariant (stationary) or time-varying (non-stationary).
- A process is stationary if the parameters of the probability model of the process are time invariant; otherwise it is non-stationary.



Strict-Sense Stationary Processes

- A random process *X(m) is* ***stationary*** *in a* ***strict sense*** if all its distributions and statistical parameters such as the mean, the variance, the power spectral composition and the higher-order moments of the process, are time-invariant.
  - $E[x(m)] = \mu_x$; mean
  - $E[x(m)x(m + k)] = r_{xx}(k)$; variance
  - $E[|X(f,m)|^2] = E[|X(f)|^2] = P_{xx}(f)$; power spectrum

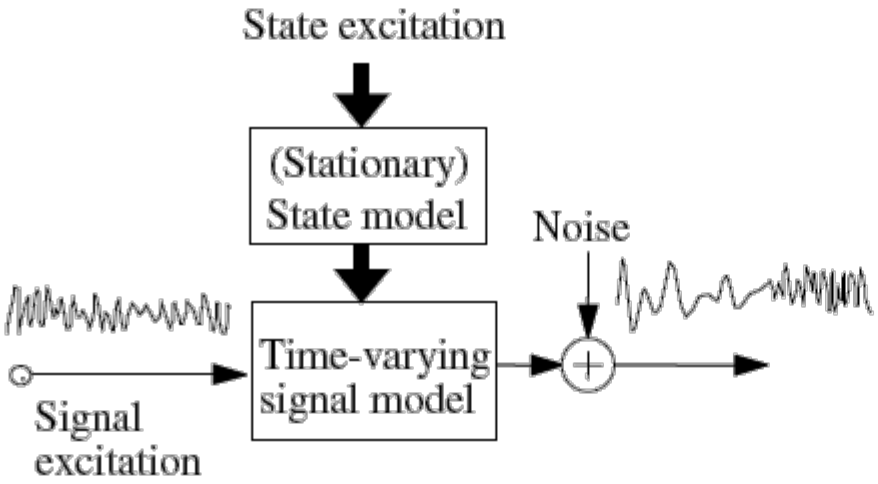
Wide-Sense Stationary Processes

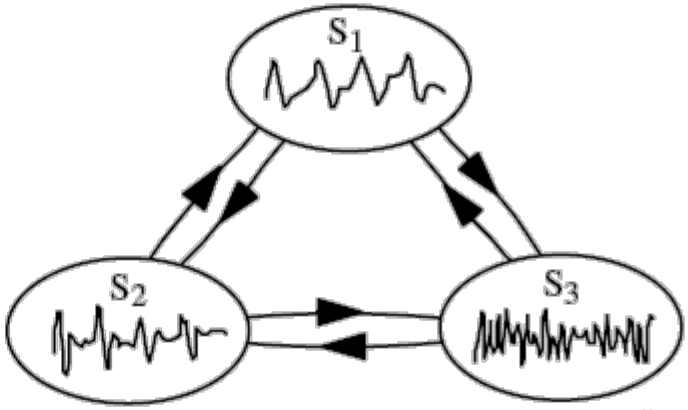
- A process is said to be ***wide sense stationary*** if the mean and the autocorrelation functions of the process are time invariant:

- $E[x(m)] = \mu_x$
- $E[x(m)x(m + k)] = r_{xx}(k)$

Non-Stationary Processes

- A random process is ***non-stationary*** if its distributions or statistics vary with time.
- Most stochastic processes such as video signals, audio signals, financial data, meteorological data, biomedical signals, etc., are nonstationary, because they are generated by systems whose environments and parameters vary over time.





Adaptive Filters

- An adaptive filter is in reality a ***nonlinear device***, in the sense that it does not obey the principle of superposition.
- Adaptive filters are commonly classified as:
  - Linear

    An adaptive filter is said to be *linear* if the estimate of quantity of interest is computed adaptively (at the output of the filter) as a *linear combination* of the available set of observations applied to the filter input.
  - Nonlinear

    Neural Networks

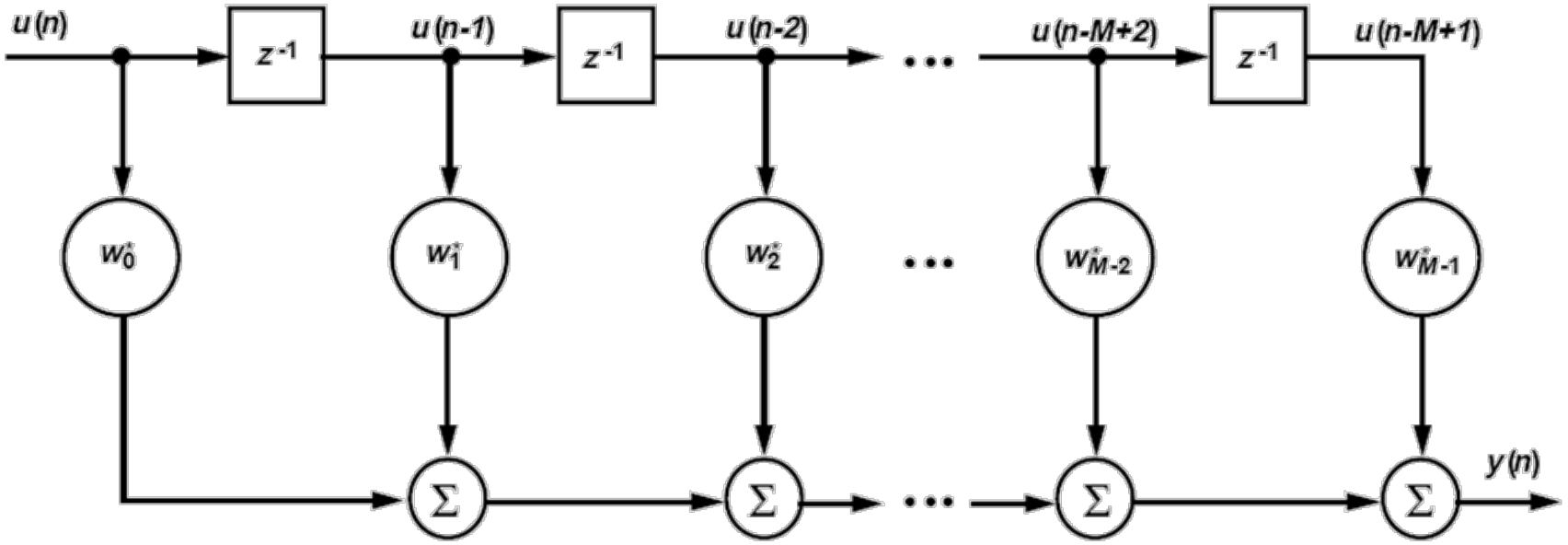
Linear Filter Structures

- The operation of a linear adaptive filtering algorithm involves two basic processes:
  - a ***filtering process*** designed to produce an output in response to a sequence of input data
  - an **adaptive process**, the purpose of which is to provide mechanism for the *adaptive control* of an *adjustable* set of parameters used in the filtering process.
- These two processes work interactively with each other.
- There are three types of filter structures with *finite memory :*
  - transversal filter,
  - lattice predictor,
  - and systolic array.

Linear Filter Structures

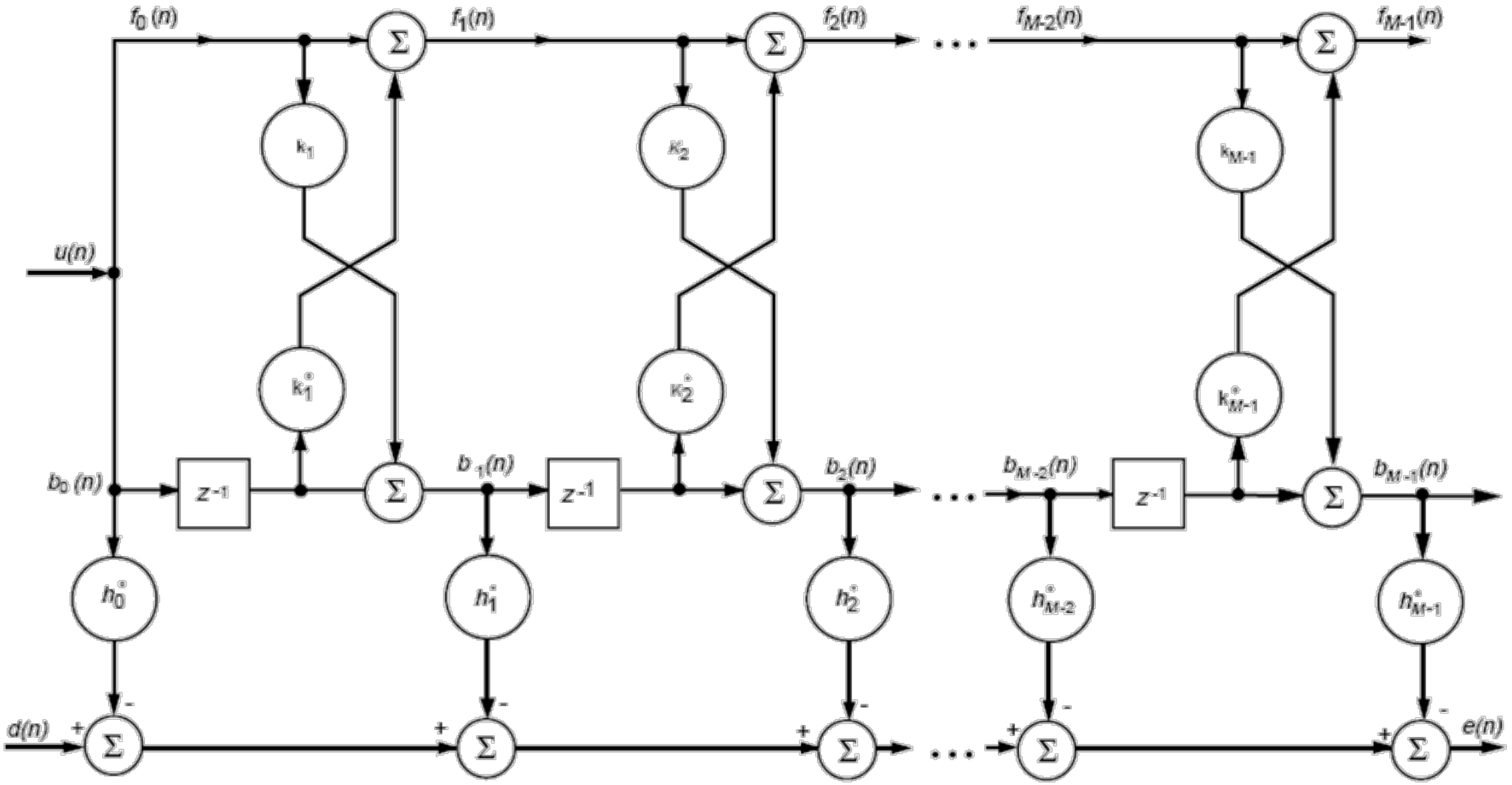
- For **stationary inputs**, the resulting solution is commonly known as the ***Wiener filter***, which is said to be optimum in the *mean-square* sense.
- A plot of the mean-square value of the error signal vs. the adjustable parameters of a linear filter is referred to as the *error-performance surface.*
- The minimum point of this surface represents the Wiener solution.
- The Wiener filter is inadequate for dealing with situations in which *non-stationarity* of the signal and/or noise is intrinsic to the problem.
- A highly successful solution to this more difficult problem is found in the ***Kalman filter***, a powerful device with a wide variety of engineering applications.

Transversal Filter



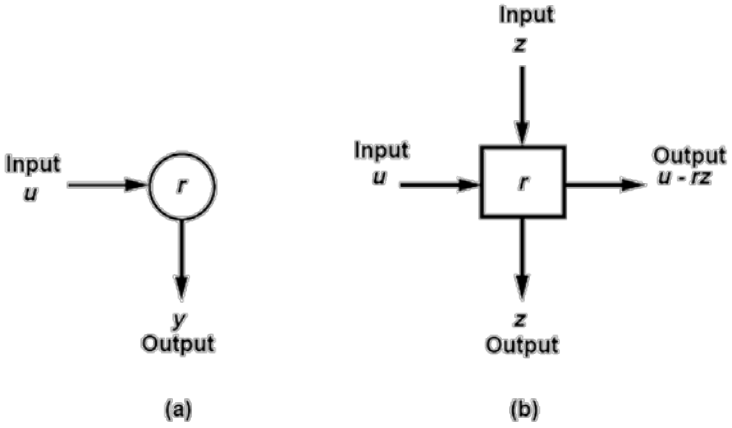
$$y(n) = \sum_{k=0}^{m-1} w_k^* u(n-k)$$

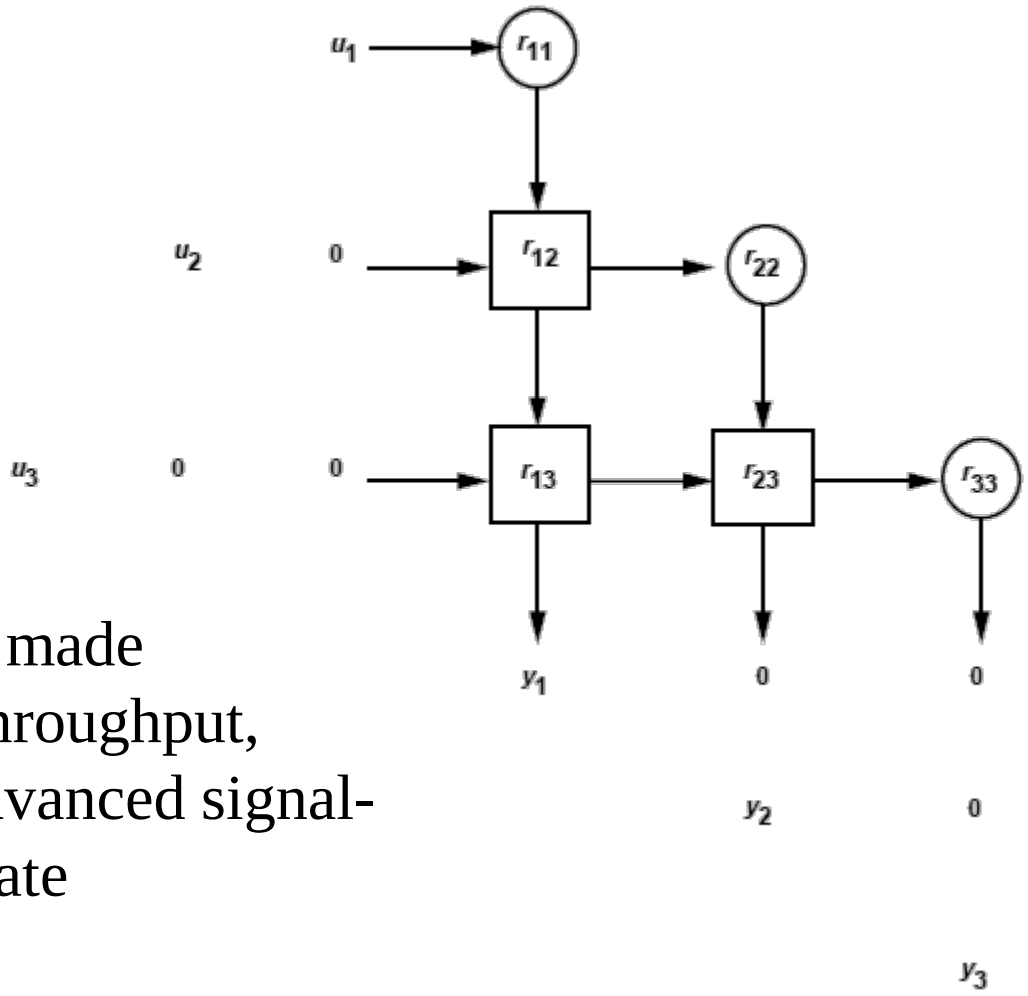
Lattice Predictor



It has the advantage of simplifying the computation

Systolic Array





The use of systolic arrays has made it possible to achieve a high throughput, which is required for many advanced signal-processing algorithms to operate in *real time*

Linear Adaptive Filtering Algorithms

- **Stochastic Gradient Approach**
  - *Least-Mean-Square* (*LMS*) *algorithm*
  - *Gradient Adaptive Lattice* (*GAL*) *algorithm*

$$\begin{pmatrix}\text{updated value}\\ \text{of tap-weight}\\ \text{vector}\end{pmatrix} = \begin{pmatrix}\text{old value}\\ \text{of tap-weight}\\ \text{vector}\end{pmatrix} + \begin{pmatrix}\text{learning-}\\ \text{rate}\\ \text{parameter}\end{pmatrix}\begin{pmatrix}\text{tap-}\\ \text{input}\\ \text{vector}\end{pmatrix}\begin{pmatrix}\text{error}\\ \text{signal}\end{pmatrix}$$

- **Least-Squares Estimation**
  - *Recursive least-squares* (*RLS*) *estimation*
    - *Standard RLS algorithm*
    - *Square-root RLS algorithms*
    - *Fast RLS algorithms*

$$\begin{pmatrix}\text{updated value}\\ \text{of the}\\ \text{state}\end{pmatrix} = \begin{pmatrix}\text{old value}\\ \text{of the}\\ \text{state}\end{pmatrix} + \begin{pmatrix}\text{Kalman}\\ \text{gain}\end{pmatrix}\begin{pmatrix}\text{innovation}\\ \text{vector}\end{pmatrix}$$

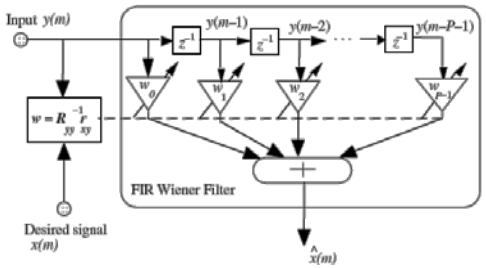
Wiener Filters

- The design of a Wiener filter requires *a priori* information about the statistics of the data to be processed.
- The filter is optimum only when the statistical characteristics of the input data match the *a priori* information on which the design of the filter is based.
- When this information is not known completely, however, it may not be possible to design the Wiener filter or else the design may no longer be optimum.

Wiener Filters: Least Square Error Estimation

- The filter input–output relation is given by:

$$\hat{x}(m) = \sum_{k=0}^{P-1} w_k\, y(m-k)$$
$$= \boldsymbol{w}^{\mathrm{T}} \boldsymbol{y}$$



- The Wiener filter error signal, *e(m) is defined* as the difference between the desired signal *x(m) and the filter output signal xˆ (m) :*

$$e(m) = x(m) - \hat{x}(m)$$
$$= x(m) - \boldsymbol{w}^{\mathrm{T}} \boldsymbol{y}$$

- error signal *e(m) for N samples of the signals x(m) and y(m):*

$$\begin{pmatrix} e(0) \\ e(1) \\ e(2) \\ \vdots \\ e(N-1) \end{pmatrix} = \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{pmatrix} - \begin{pmatrix} y(0) & y(-1) & y(-2) & \cdots & y(1-P) \\ y(1) & y(0) & y(-1) & \cdots & y(2-P) \\ y(2) & y(1) & y(0) & \cdots & y(3-P) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y(N-1) & y(N-2) & y(N-3) & \cdots & y(N-P) \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_{P-1} \end{pmatrix}$$

$$\boldsymbol{e} = \boldsymbol{x} - \boldsymbol{Y}\boldsymbol{w}$$

Wiener Filters: Least Square Error Estimation

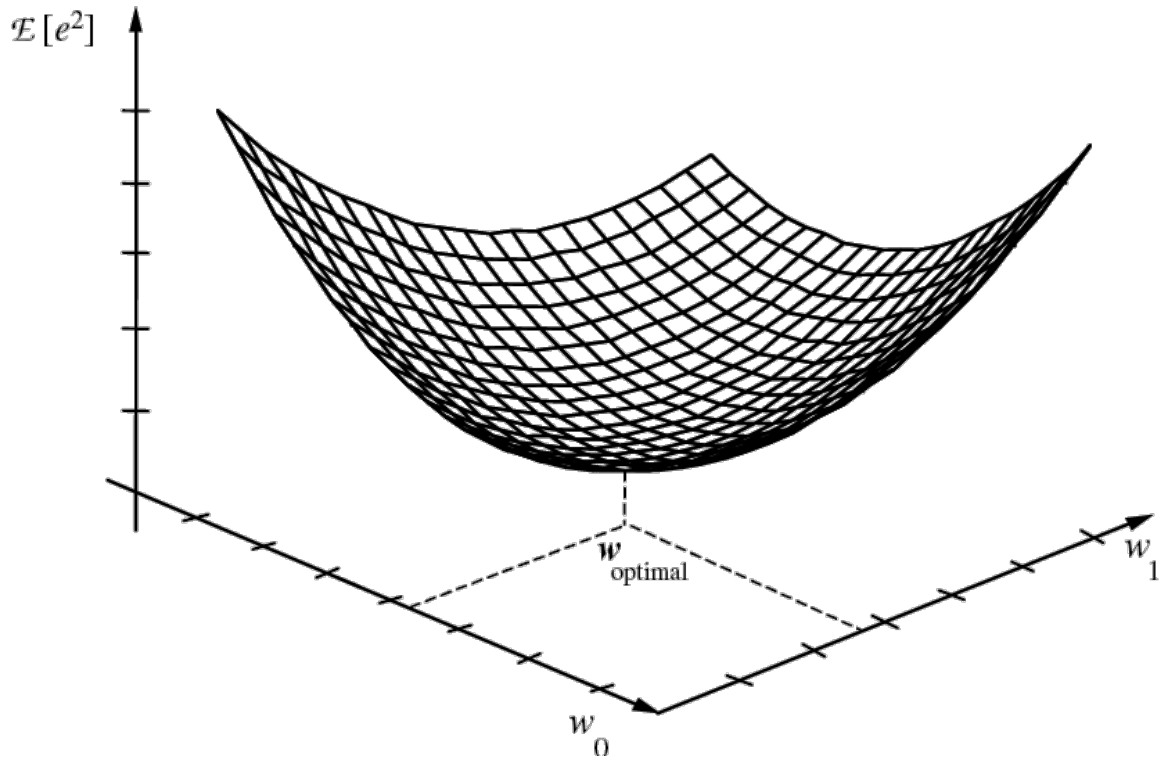
- The Wiener filter coefficients are obtained by minimising an average squared error function $E[e^2(m)]$ *with respect to the filter* coefficient vector ***w***

$$\begin{aligned}\mathcal{E}[e^2(m)] &= \mathcal{E}[(x(m)-\boldsymbol{w}^{\mathrm{T}}\boldsymbol{y})^2] \\ &= \mathcal{E}[x^2(m)]-2\boldsymbol{w}^{\mathrm{T}}\mathcal{E}[\boldsymbol{y}x(m)]+\boldsymbol{w}^{\mathrm{T}}\mathcal{E}[\boldsymbol{y}\boldsymbol{y}^{\mathrm{T}}]\boldsymbol{w} \\ &= r_{xx}(0)-2\boldsymbol{w}^{\mathrm{T}}\boldsymbol{r}_{yx}+\boldsymbol{w}^{\mathrm{T}}\boldsymbol{R}_{yy}\boldsymbol{w}\end{aligned}$$

- $R_{yy}=E\,[y(m)y^T(m)]$ is the autocorrelation matrix of the input signal
- $r_{xy}=E\,[x(m)y(m)]$ is the cross-correlation vector of the input and the desired signals

Wiener Filters: Least Square Error Estimation

- For example, for a filter with only two coefficients (*w0, w1), the mean square error function is a* bowl-shaped surface, with a single minimum point



Wiener Filters: Least Square Error Estimation

- The gradient vector is defined as

$$\frac{\partial}{\partial \boldsymbol{w}} = \left[\frac{\partial}{\partial w_0}, \frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}, \ldots, \frac{\partial}{\partial w_{P-1}} \right]^{\mathrm{T}}$$

- Where the gradient of the mean square error function with respect to the filter coefficient vector is given by

$$\frac{\partial}{\partial \boldsymbol{w}} \mathcal{E}[e^2(m)] = -2\mathcal{E}[x(m)\,\boldsymbol{y}(m)] + 2\boldsymbol{w}^{\mathrm{T}} \mathcal{E}[\boldsymbol{y}(m)\,\boldsymbol{y}^{\mathrm{T}}(m)]$$

$$= -2\boldsymbol{r}_{yx} + 2\boldsymbol{w}^{\mathrm{T}} \boldsymbol{R}_{yy}$$

- The minimum mean square error Wiener filter is obtained by setting equation to zero

$$\boldsymbol{R}_{yy}\,\boldsymbol{w} = \boldsymbol{r}_{yx} \qquad \boldsymbol{w} = \boldsymbol{R}_{yy}^{-1}\,\boldsymbol{r}_{yx}$$

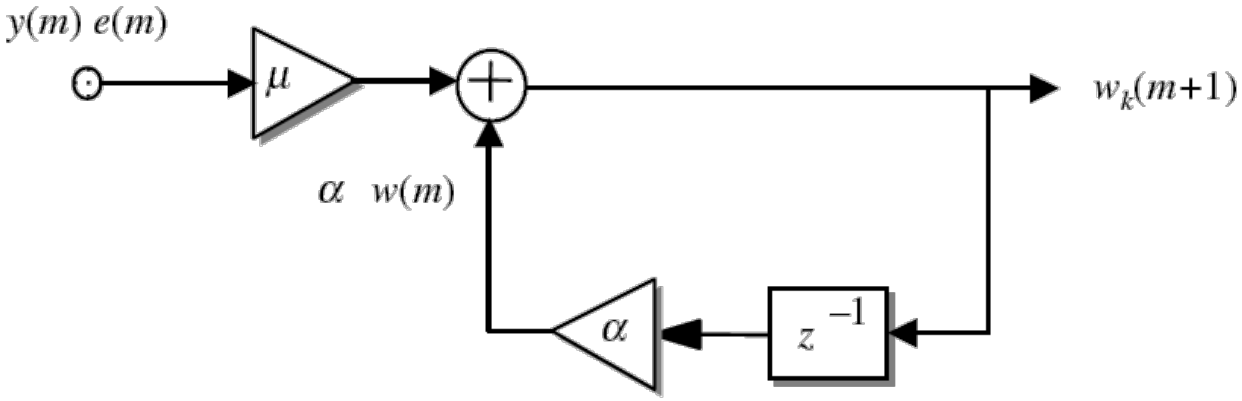
Wiener Filters: Least Square Error Estimation

- The calculation of the Wiener filter coefficients requires the ***autocorrelation*** matrix of the ***input signal*** and the ***crosscorrelation*** vector of the **input and the desired signals.**

- The optimum **w** value is $\boldsymbol{w_o = R_{yy}^{-1} r_{yx}}$

The LMS Filter

- A computationally simpler version of the gradient search method is the least mean square (LMS) filter, in which the ***gradient of the mean square error*** *is* substituted with the ***gradient of the instantaneous squared error function.***



- Note that the feedback equation for the time update of the filter coefficients is essentially a recursive (infinite impulse response) system with input μ[y(m)e(m)] and its poles at α.

The LMS Filter

- The LMS adaptation method is defined as

$$\boldsymbol{w}(m+1) = \boldsymbol{w}(m) + \mu \left(-\frac{\partial\, e^2(m)}{\partial\, \boldsymbol{w}(m)} \right)$$

- The instantaneous gradient of the squared error can be expressed as

$$\begin{aligned} \frac{\partial e^2(m)}{\partial \boldsymbol{w}(m)} &= \frac{\partial}{\partial \boldsymbol{w}(m)} [x(m) - \boldsymbol{w}^{\mathrm{T}}(m)\boldsymbol{y}(m)]^2 \\ &= -2\boldsymbol{y}(m)[x(m) - \boldsymbol{w}^{\mathrm{T}}(m)\boldsymbol{y}(m)]^2 \\ &= -2\boldsymbol{y}(m)e(m) \end{aligned}$$

- Substituting this equation into the recursion update equation of the filter parameters, yields the LMS adaptation equation

$$\boldsymbol{w}(m+1) = \boldsymbol{w}(m) + \mu\, [\boldsymbol{y}(m)e(m)]$$

The LMS Filter

- The main advantage of the LMS algorithm is its simplicity both in terms of the memory requirement and the computational complexity which is *O(P)*, where P is the filter length
- **Leaky LMS Algorithm**
  - The stability and the adaptability of the recursive LMS adaptation can improved by introducing a so-called leakage factor α as

$$w(m+1) = \alpha . w(m) + \mu . [y(m) . e(m)]$$

  - When the parameter α<1, the effect is to introduce more stability and accelerate the filter adaptation to the changes in input signal characteristics.

LMS Algorithm

Notations:

$\mathbf{u}(n)$	= tap-input vector at time n
	= $[u(n), u(n-1), \ldots, u(n-M+1)]^T$
M	= number of tap inputs
$d(n)$	= desired response at time n
$\hat{\mathbf{w}}(n)$	= $[w_0(n), w_1(n), \ldots, w_{M-1}(n)]^T$
	= tap-weight vector at time n
$y(n)$	= actual response of the tapped-delay line filter
	= $\hat{\mathbf{w}}^H(n)\mathbf{u}(n)$, where superscript H denotes Hermitian transposition
$e(n)$	= error signal
	= $d(n) - y(n)$

Parameters:

M	= number of taps
m	= step-size parameter

$$0 < \mu < \frac{2}{\text{tap-input power}}$$

$$\text{tap-input power} = \sum_{k=0}^{M-1} E[|u(n-k)|^2]$$

Initialization:

If prior knowledge on the tap-weight vector $\hat{\mathbf{w}}(n)$ is available, use it to select an appropriate value for $\hat{\mathbf{w}}(0)$. Otherwise, set $\hat{\mathbf{w}}(0) = \mathbf{0}$.

Date:

Given: $\mathbf{u}(n) = M \times 1$ tap-input vector at time n

$d(n)$ = desired response at time n

To be computed: $\hat{\mathbf{w}}(n+1)$ = estimate of tap-weight vector at time $n+1$

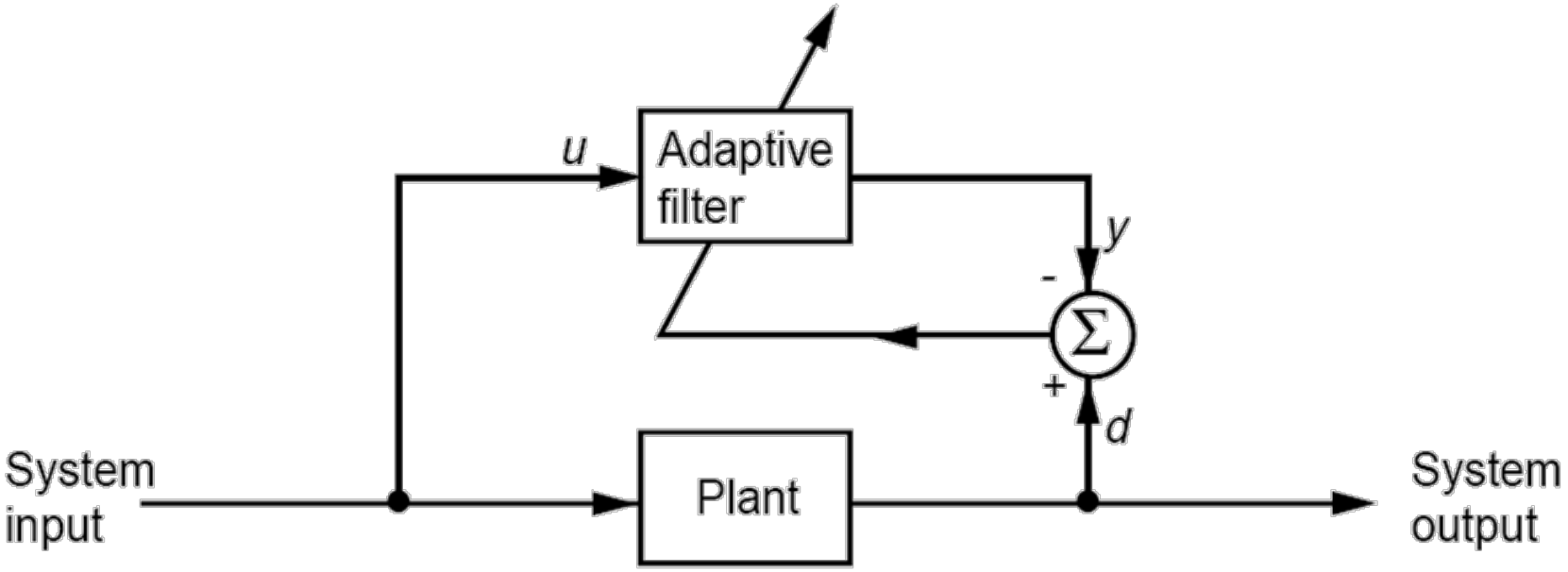
Computation: For $n = 0, 1, 2, \ldots$, compute

$$e(n) = d(n) - \hat{\mathbf{w}}^H(n)\mathbf{u}(n)$$

$$\hat{\mathbf{w}}(n+1) = \hat{\mathbf{w}}(n) = \mu\mathbf{u}(n)e^*(n)$$

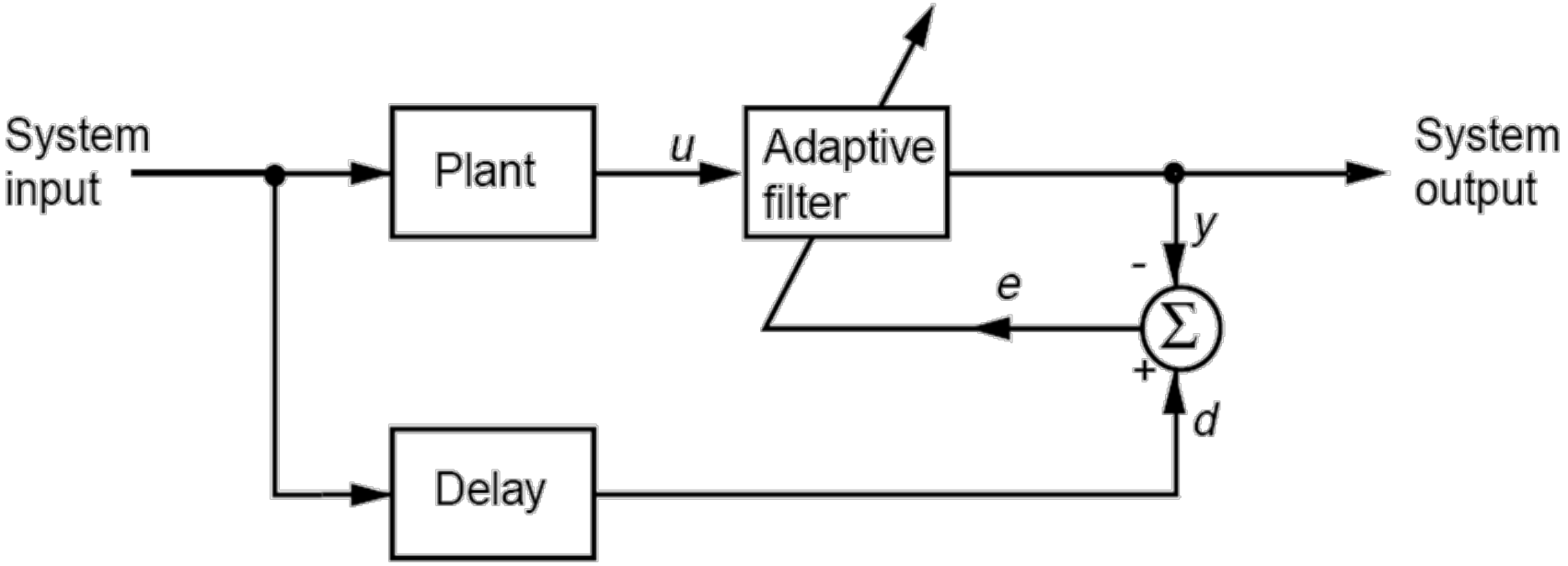
```
Wk=zeros(1,L+1);              % Vector Inicial de Pesos
yk=zeros(size(xk));           % Señal de salida inicial del FIR
ek=zeros(size(xk));           % Señal inicial de error


for i=L+1:N-1
  for n=1:L+1
    xk_i(1,n)=xk(i+1-n);      % Vector x i-ésimo
  end
  yk(i)=xk_i*Wk';             % señal a la salida del FIR
  ek(i)=dk(i)-yk(i);          % Señal de error
  Wk=Wk+2*mu*ek(i)*xk_i;      % Vector de pesos i-ésimo
end
```

Identification



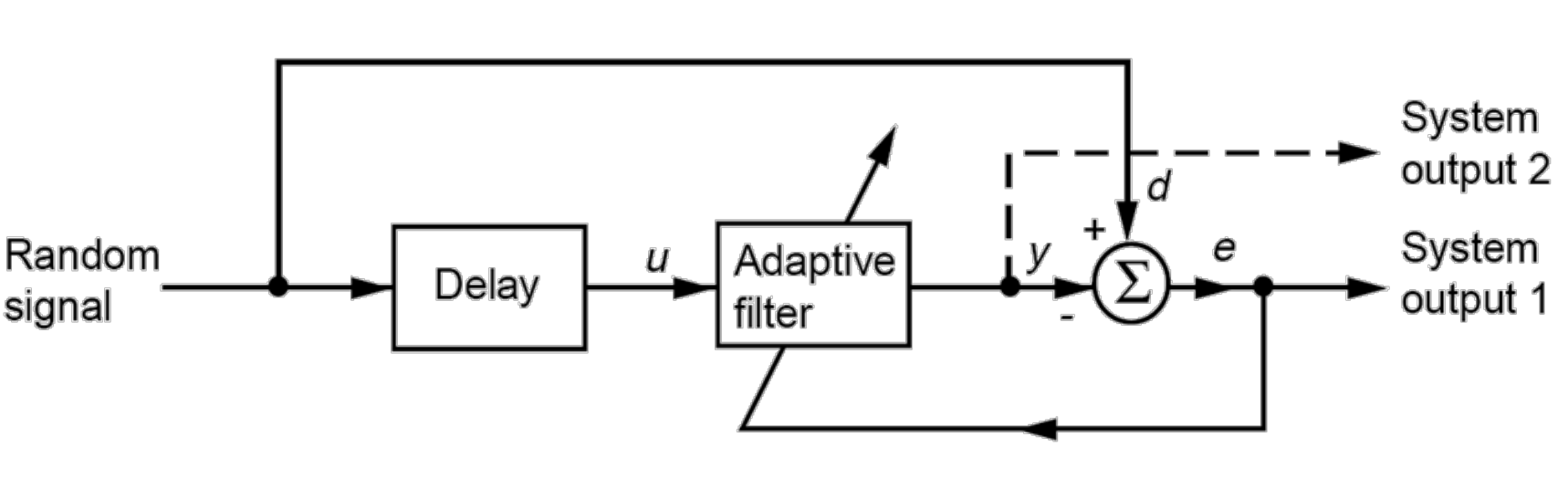
- System identification
- Layered earth modeling

Inverse modeling



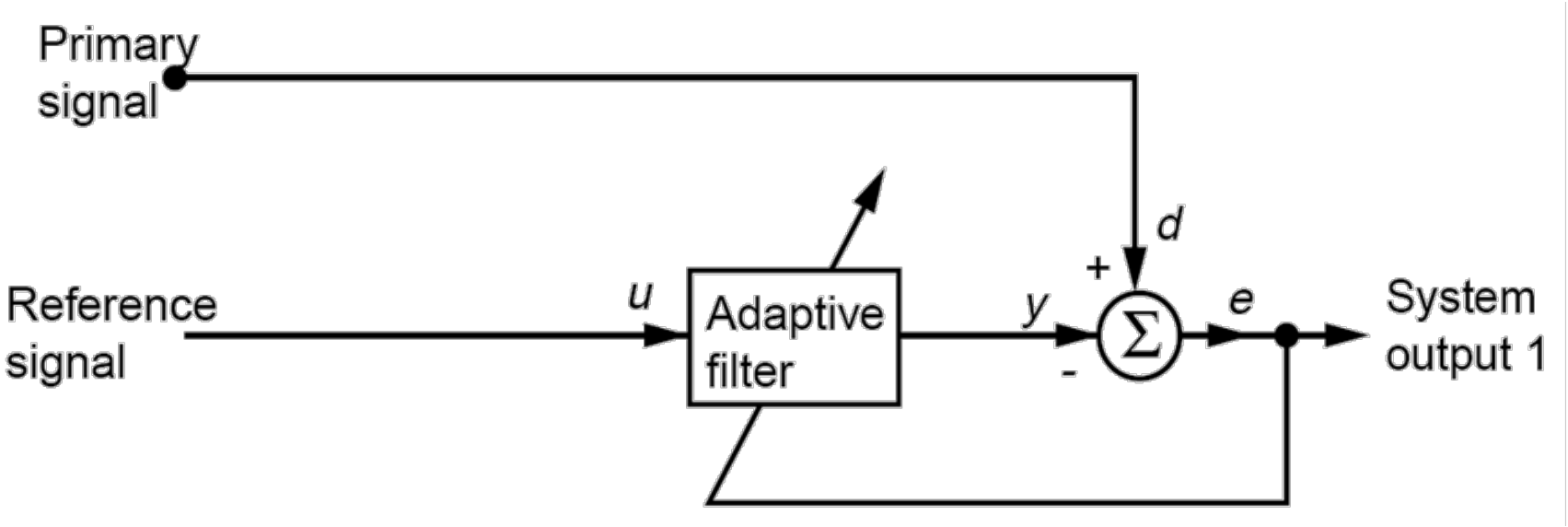
- Predictive deconvolution
- Adaptive equalization
- Blind equalization

Prediction



- Linear predictive coding
- Adaptive differential pulse-code modulation
- Autoregressive spectrum analysis
- Signal detection

Interference canceling

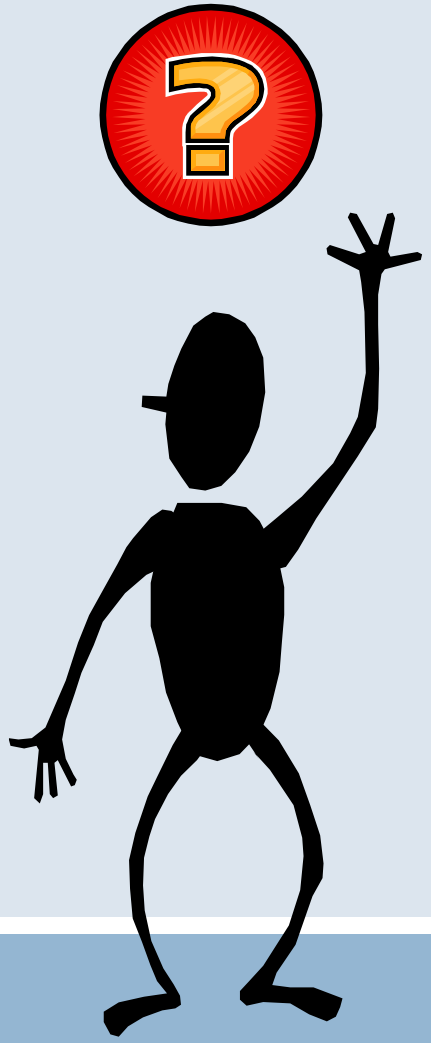


- Adaptive noise canceling
- Echo cancelation
- Adaptive beamforming

Recommended bibliography

- Saeed V. Vaseghi, *Advanced Digital Signal Processing and Noise Reduction*, Second Edition. John Wiley & Sons Ltd.
  - Ch 3: Probabilistic Models
  - Ch 6: Wiener Filters
  - Ch 7: Adaptive Filters
  - Ch 8: Linear Prediction Models
- Stergios Stergiopoulos, *Advanced Signal Processing Handbook.* CRC Press LLC, 2001
  - Ch 2: Adaptive Systems for Signal Process - Simon Haykin
- B Farhang-Boroujeny. *Adaptive Filters. Theory and Applications*. John Wiley & Sons.

- **NOTE**: Many images used in this presentation were extracted from the recommended bibliography.



Questions?

Thank you!