

REAL TIME DIGITAL SIGNAL PROCESSING

Laplace Transform

A brief overview

The fundamental process of using the Laplace transform

- A time-domain differential equation is written that describes the input/output relationship of a physical system (and we want to find the output function that satisfies that equation with a given input).
- The differential equation is Laplace transformed, converting it to an algebraic equation.
- Standard algebraic techniques are used to determine the desired output function's equation in the Laplace domain.
- The desired Laplace output equation is, then, inverse Laplace transformed to yield the desired time-domain output function's equation.

Marquis Pierre Simon de Laplace's (1749–1827)

Laplace transform

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt \quad , \quad s = \sigma + j\omega$$

- We can say that LT requires us to multiply, point for point, the function $f(t)$ by the complex function e^{-st} for a given value of s .
- After that, we find the area under the curve of the function $f(t)e^{-st}$ by summing all the products.
- That area is a complex number.
- We were to go through this process for all values of s , we'd have a full description of $F(s)$ for every value of s .

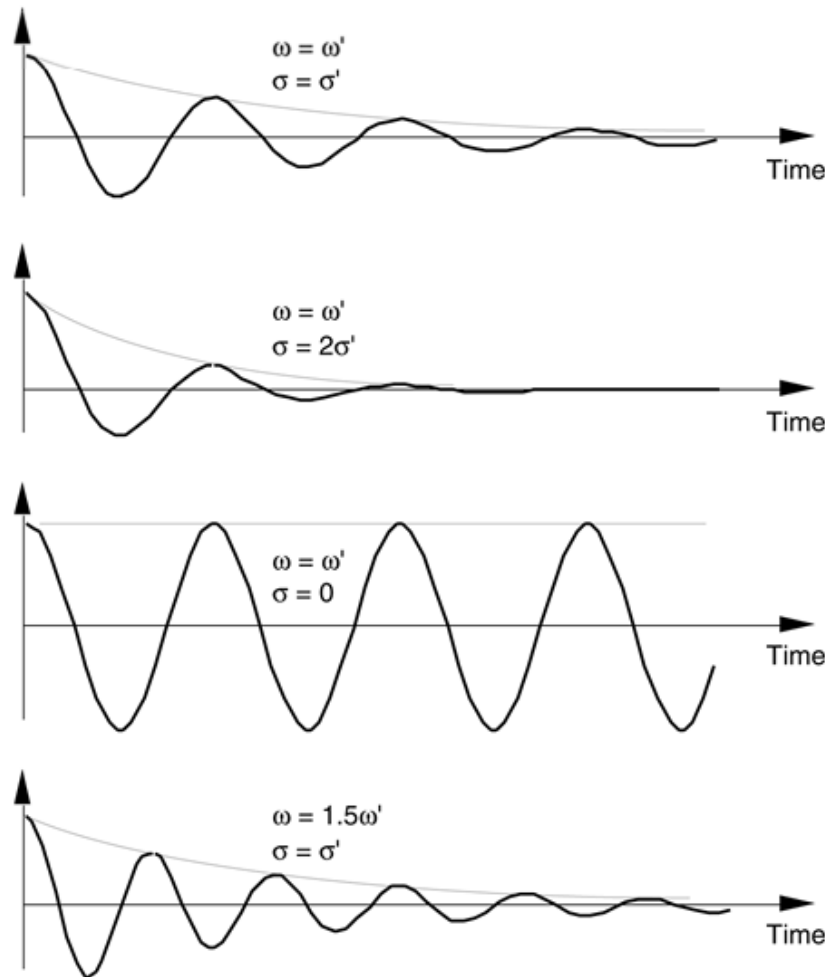
Laplace transform (II)

- The complex value of LT for a particular value of s is a correlation of $f(t)$ and a damped complex e^{-st} sinusoid whose frequency is ω and whose damping factor is σ .

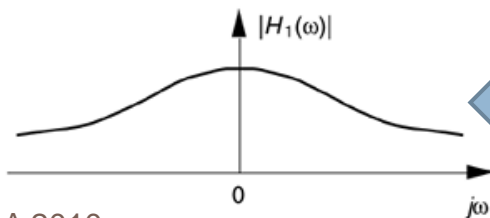
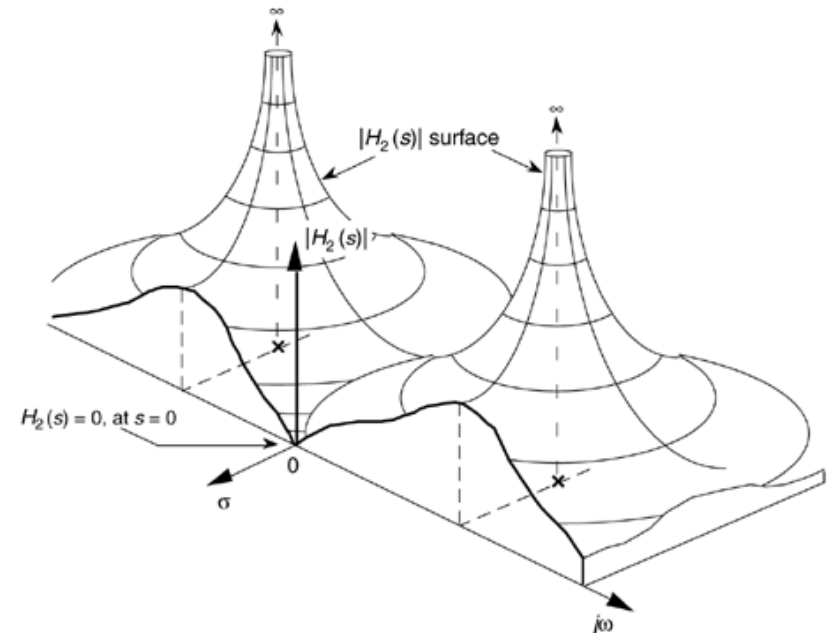
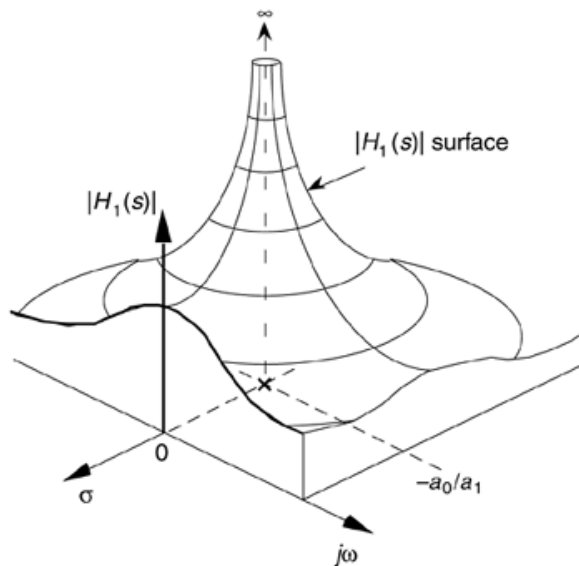
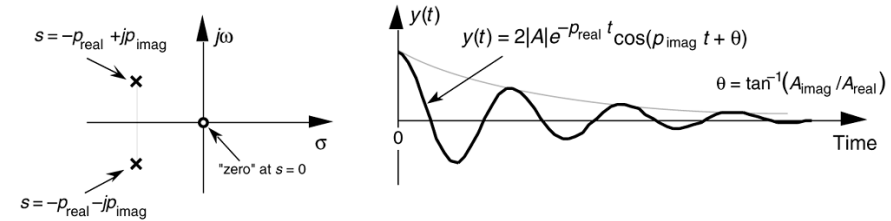
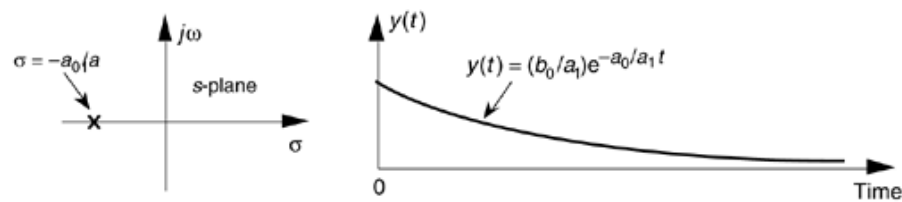
$$e^{-st} = e^{-(\sigma + j\omega)t} = e^{-\sigma t} e^{-j\omega t} = \frac{e^{-j\omega t}}{e^{\sigma t}} = \frac{\cos(\omega t)}{e^{\sigma t}} - j \frac{\sin(\omega t)}{e^{\sigma t}}$$

- Laplace transform is a more general case of the Fourier transform. ($\sigma=0$)

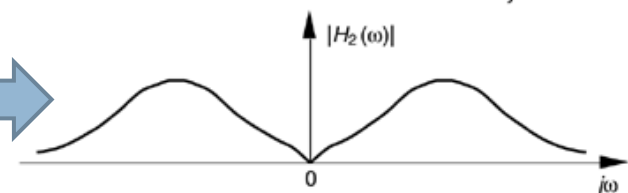
Real part (cosine) of various e^{-st} functions



Poles and Zeros on the s-Plane



Frequency
Magnitude
Response



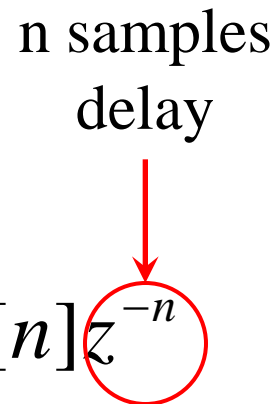
Z Transform

Introduction

Z Transform

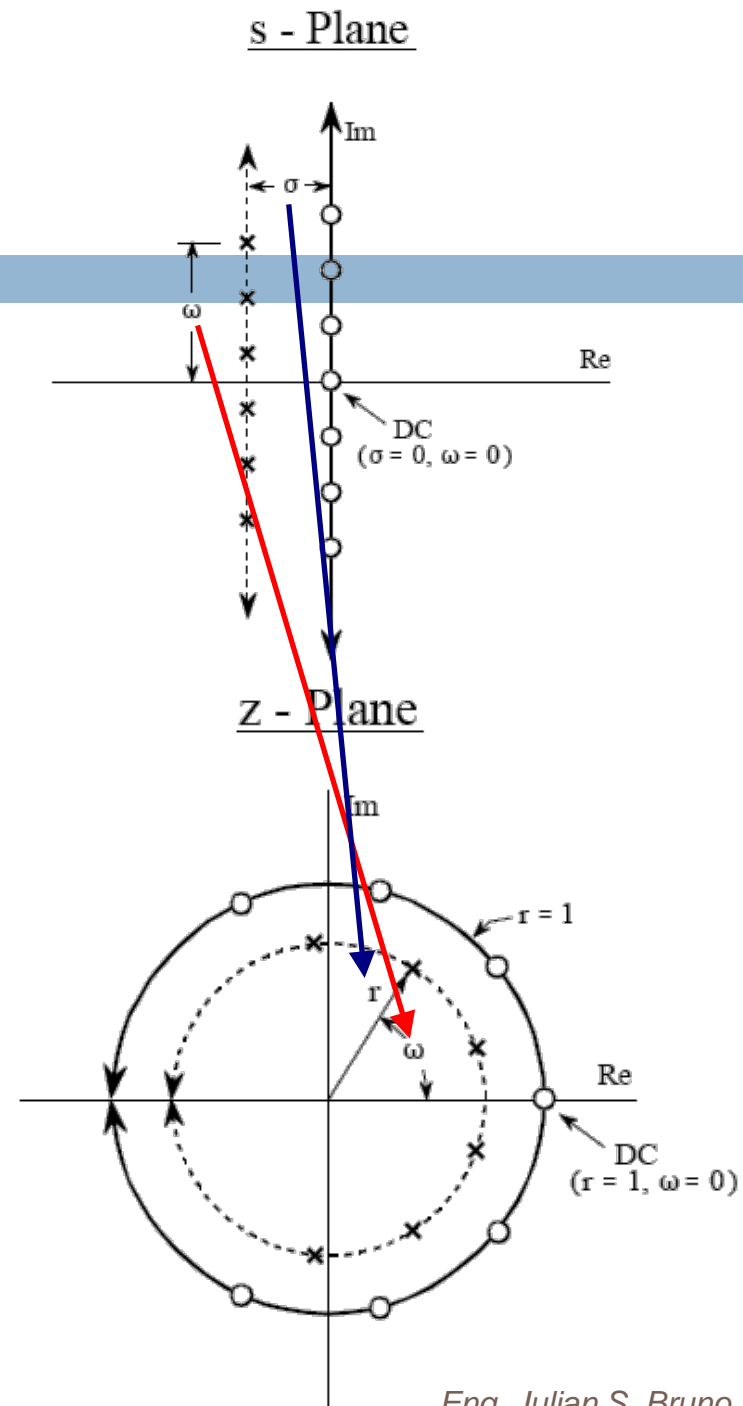
- The Z transform is quite a trivial algorithm. Each sample multiplied by the complex variable z at the power equal to its delay.
- z is a complex variable with a modulus 'r' and an argument ' ω ' (frequency).
- The inverse transform is done (typically) by simple fraction expansion method and a table of transforms.
- In real time applications, systems are causal, thus Z transform is always unillateral being 0 the lower summation limit.

n samples delay

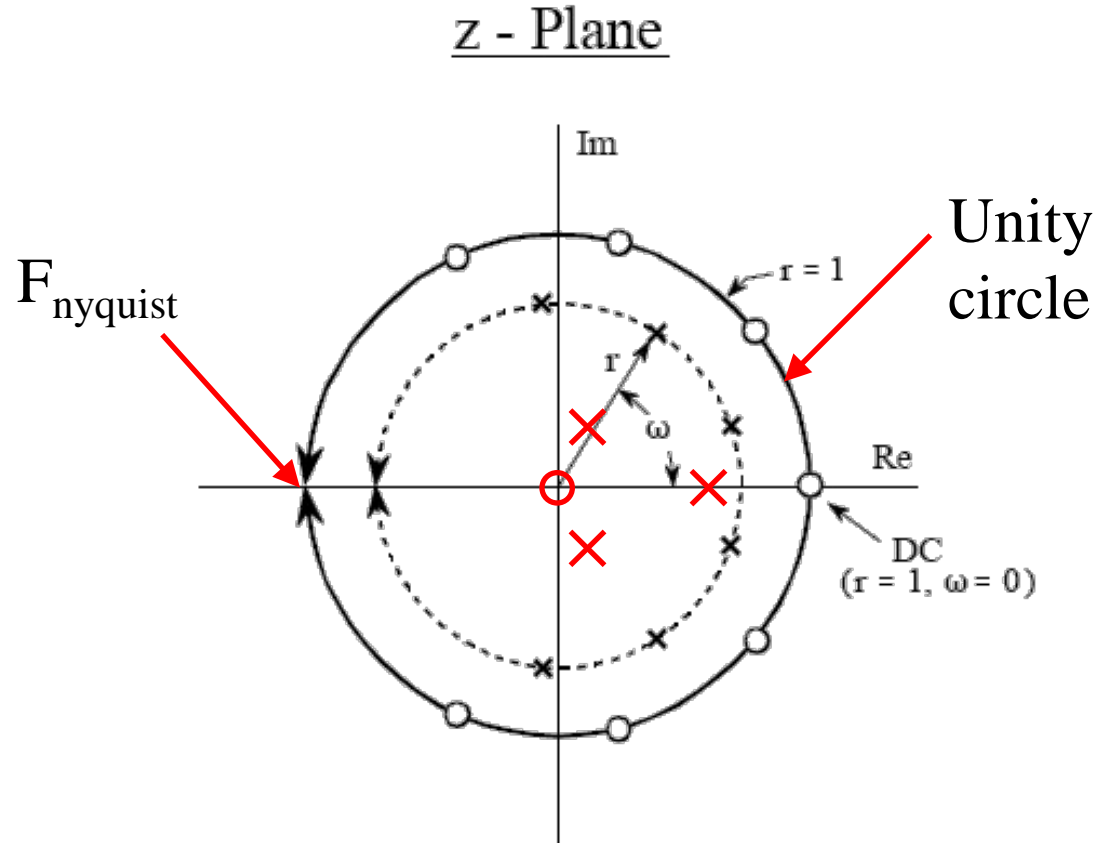
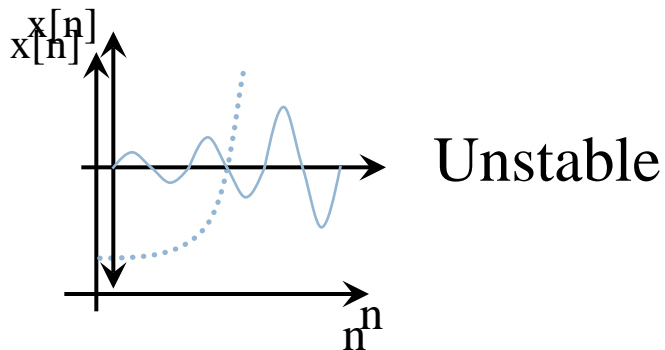
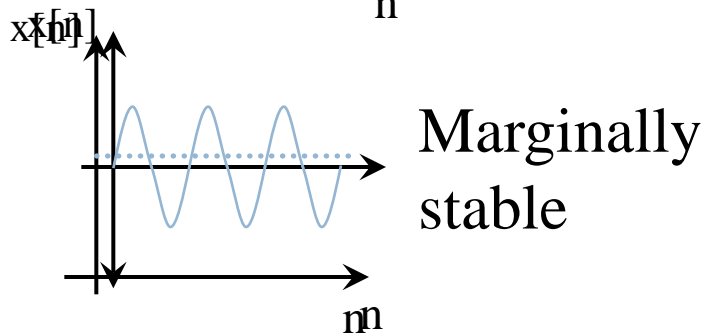
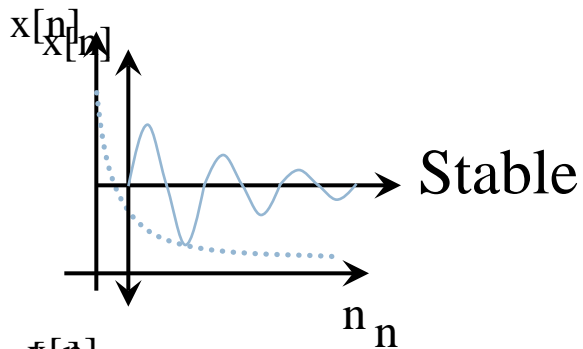

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$
$$z = r e^{j\omega}$$

Z Transform

- Z transform is the discrete counterpart of Laplace transform.
- A vector in the Z plane have a Frequency equal to its argument and a damping equal to his modulo.
- It is used to show the behavior of digital systems.
- Similar to the continuous domain, DFT is a particular case of Z transform. **Which case?**

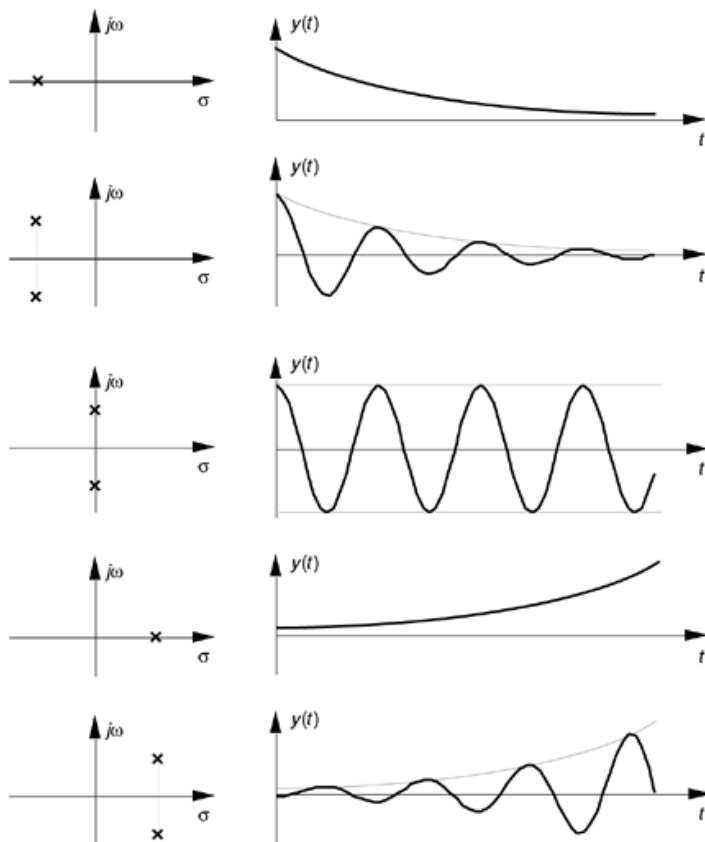


Z plane important places

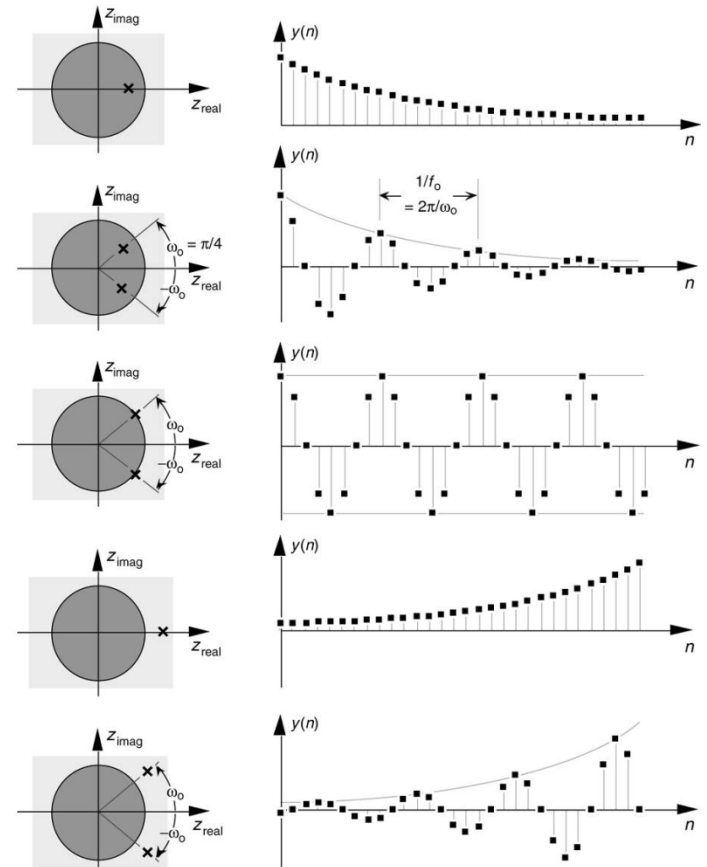


Pole locations and Time-domain impulse responses

S-plane



Z-plane



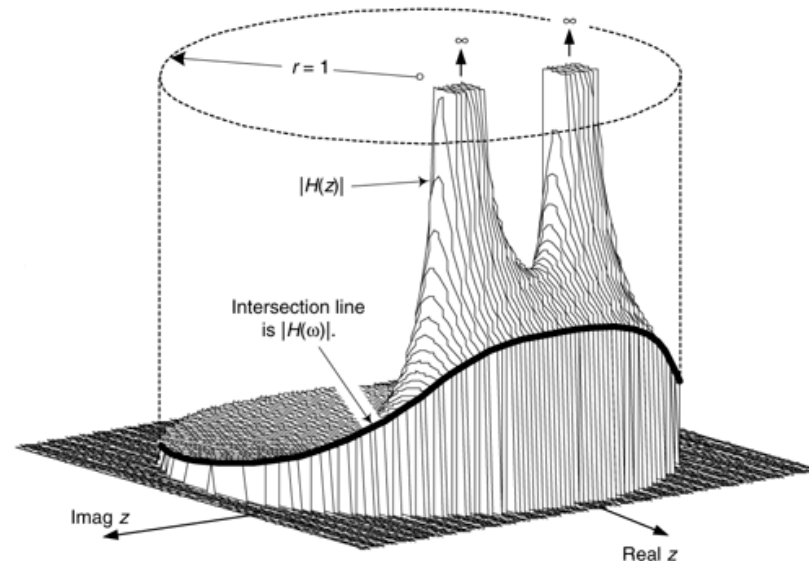
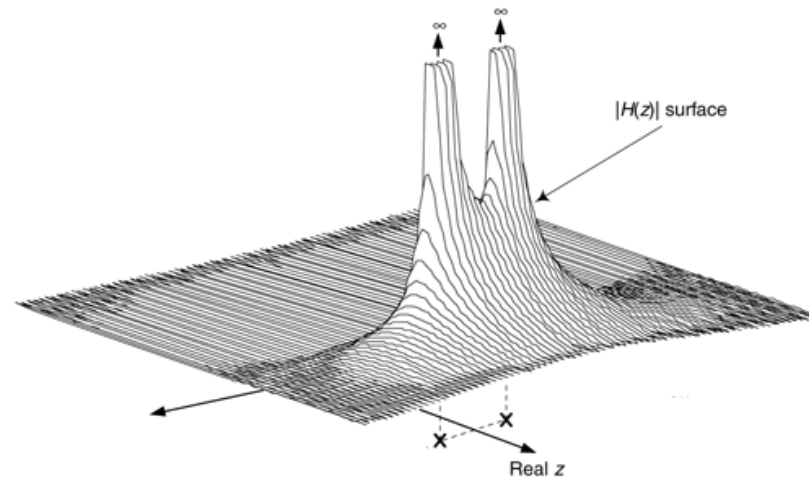
Particular Case – Fourier Transform

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \quad ; \quad z = re^{j\omega}$$

$$H(re^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]r^{-n}(e^{-j\omega n})$$

when $|z|=1, z = e^{j\omega}$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$



Z Plane System representation

Difference
equation

$$y[n] = a_0x[n] + a_1x[n-1] + a_2x[n-2] + \dots + b_1y[n-1] + b_2y[n-2] + b_3y[n-3] + \dots$$

Transfer Function

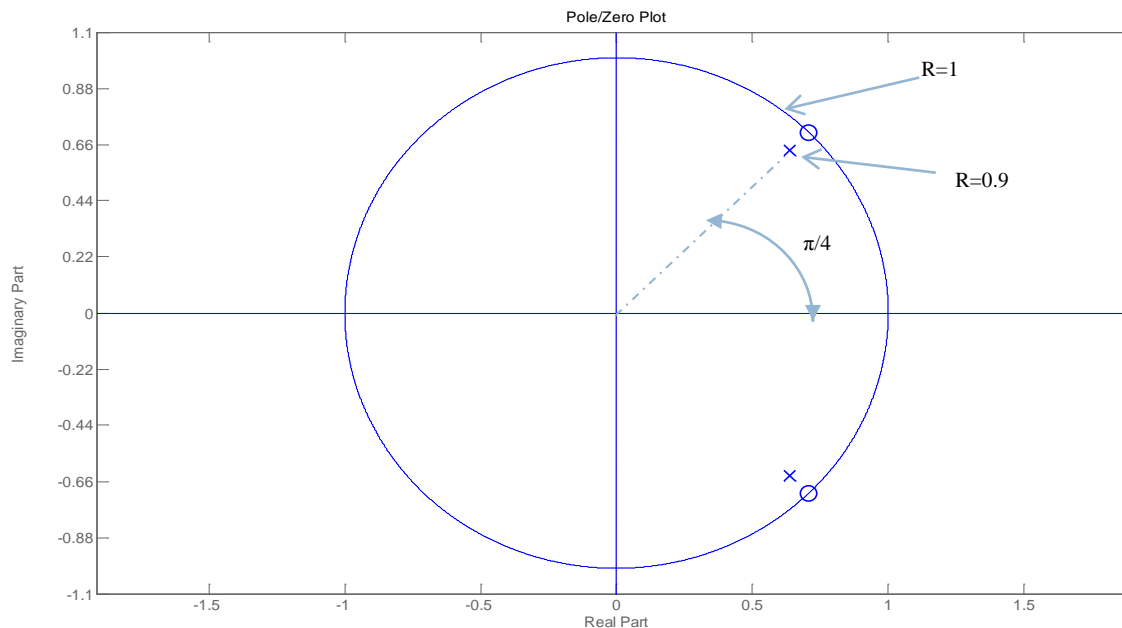
$$H[z] = \frac{a_0 + a_1z^{-1} + a_2z^{-2} + a_3z^{-3} + \dots}{1 - b_1z^{-1} - b_2z^{-2} - b_3z^{-3} - \dots}$$

$$H[z] = \frac{(z - z_1)(z - z_2)(z - z_3)\dots}{(z - p_1)(z - p_2)(z - p_3)\dots}$$

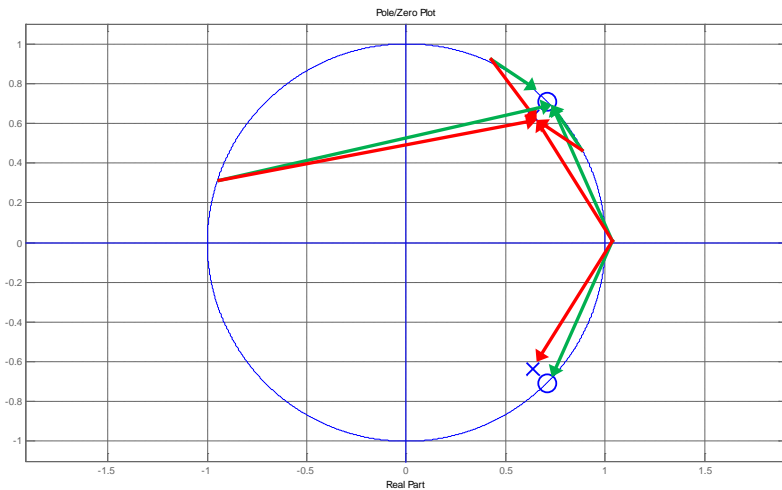
- Difference equation form is useful to **implement** the system (i.e. DSP, Matlab, etc.)
- Transfer function is useful to **design and analyze** system's behavior by zero/pole placement/location.

Example: A notch filter

- A notch filter is a system that rejects only a particular frequency.
- This is equivalent to place a zero in this particular frequency, and a pole very close to the zero.

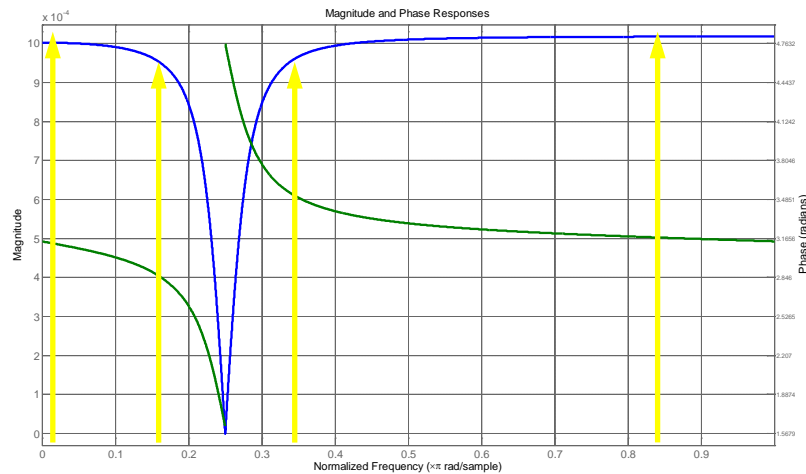


Frequency response of a notch filter



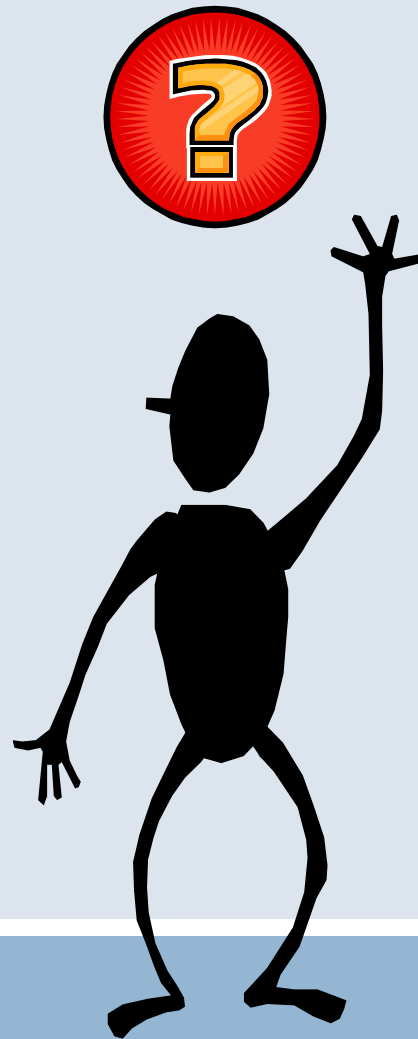
Modulo:

$$|H[z]| = \frac{\prod |v_{zero}|}{\prod |v_{pole}|}$$



Phase:

$$Arg(H[z]) = Arg(v_{zero}) - Arg(v_{pole})$$



Questions?

Thank you!