Digital Processing Laboratory
Laboratorio de Procesamiento Digital

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Real Time Digital Signal Processing

UTN-FRBA
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www.electron.frba.utn.edu.ar/dplab

Laplace Transform

A brief overview

The fundamental process of using the Laplace transform

- A time-domain differential equation is written that describes the input/output relationship of a physical system (and we want to find the output function that satisfies that equation with a given input).
- The differential equation is Laplace transformed, converting it to an algebraic equation.
- Standard algebraic techniques are used to determine the desired output function's equation in the Laplace domain.
- The desired Laplace output equation is, then, inverse Laplace transformed to yield the desired time-domain output function's equation.

Marquis Pierre Simon de Laplace's (1749–1827)

Laplace transform

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt \quad , \quad s = \sigma + j\omega$$

- We can say that LT requires us to multiply, point for point, the function f(t) by the complex function e^{-st} for a given value of s.
- After that, we find the area under the curve of the function $f(t)e^{-st}$ by summing all the products.
- That area is a complex number.
- We were to go through this process for all values of s, we'd have a full description of F(s) for every value of s.

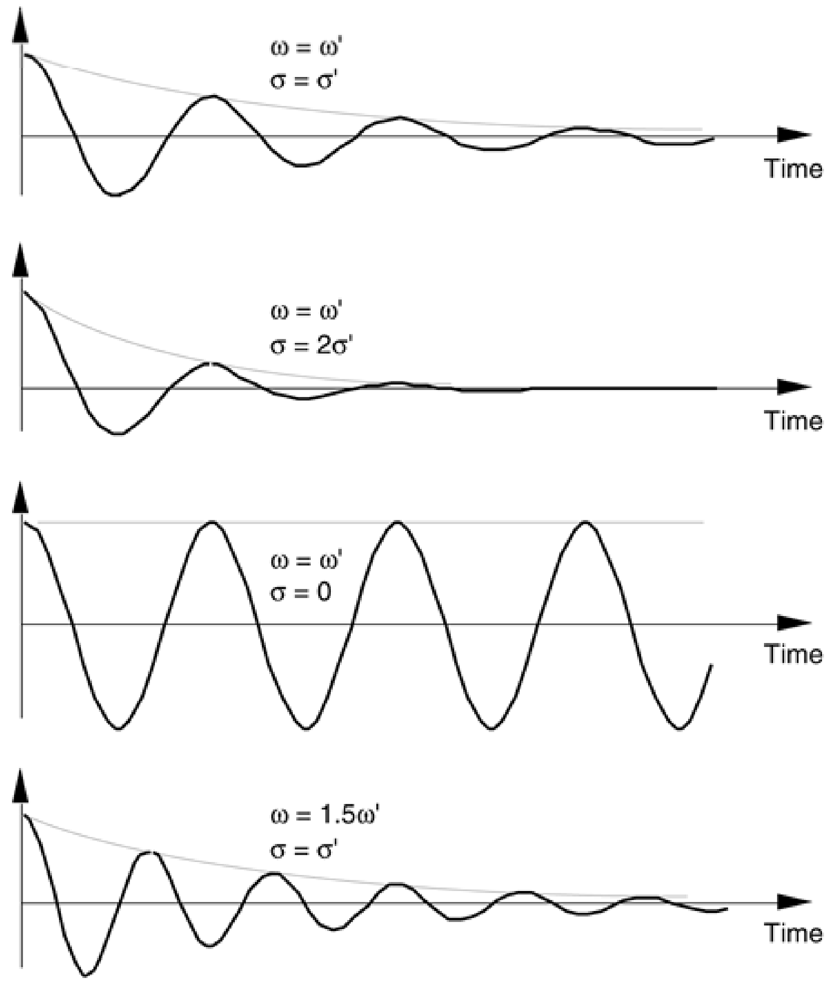
Laplace transform (II)

- The complex value of LT for a particular value of s is a correlation of f(t) and a damped complex e^{-st} sinusoid whose frequency is ω and whose damping factor is σ.

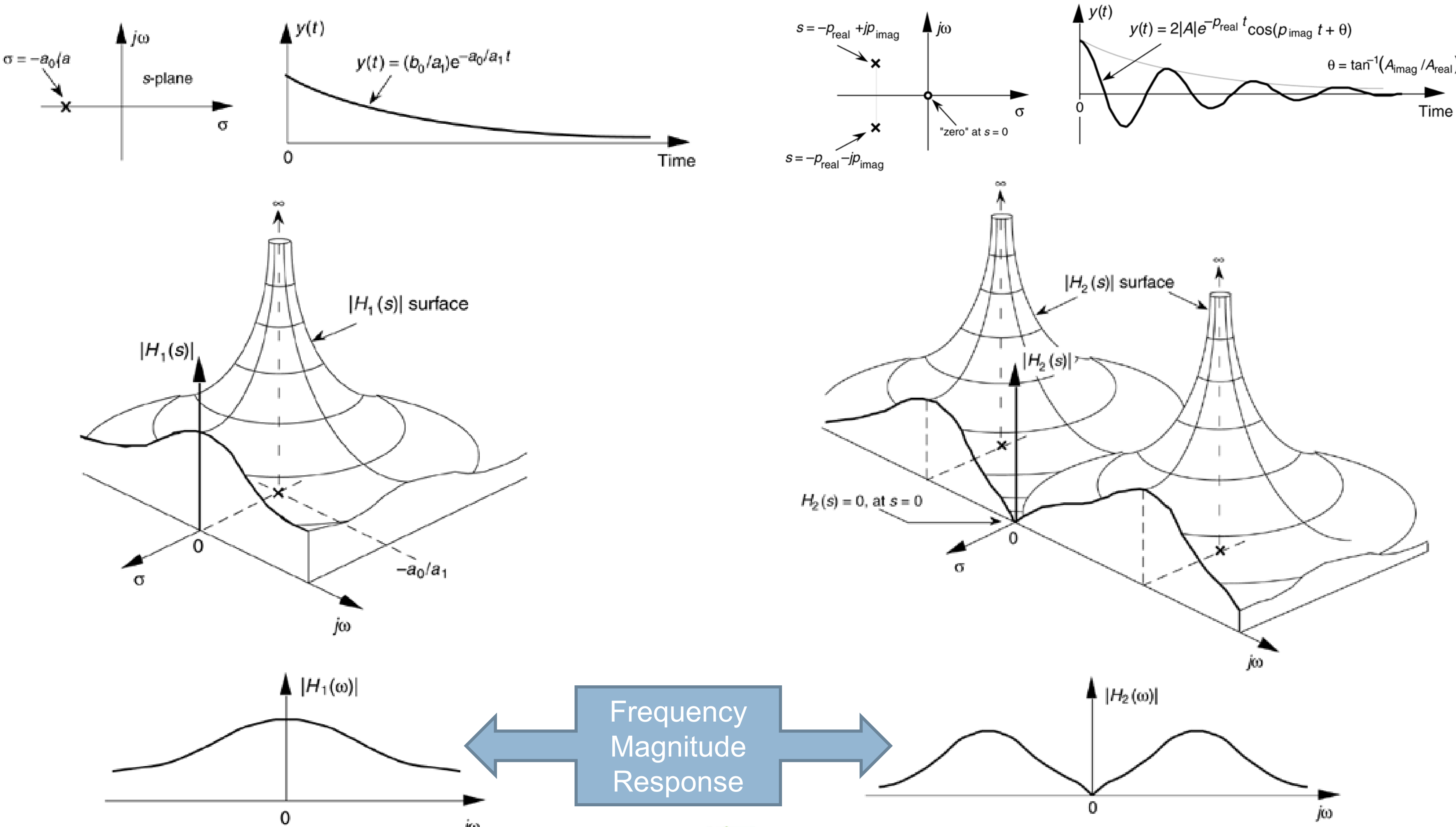
$$e^{-st} = e^{-(\sigma + j\omega)t} = e^{-\sigma t} e^{-j\omega t} = \frac{e^{-j\omega t}}{e^{\sigma t}} = \frac{\cos(\omega t)}{e^{\sigma t}} - j\frac{\sin(\omega t)}{e^{\sigma t}}$$

- Laplace transform is a more general case of the Fourier transform. (σ=0)

Real part (cosine) of various e^{-st} functions



Poles and Zeros on the s-Plane



Z Transform

Introduction

Z Transform

- The Z transform is quite a trivial algorithm. Each sample multiplied by the complex variable z at the power equal to its delay.
- z is a complex variable with a modulus ‘r’ and an argument ‘ω’ (frequency).
- The inverse transform is done (typically) by simple fraction expansion method and a table of transforms.
- In real time applications, systems are causal, thus Z transform is always unillateral being 0 the lower summation limit.

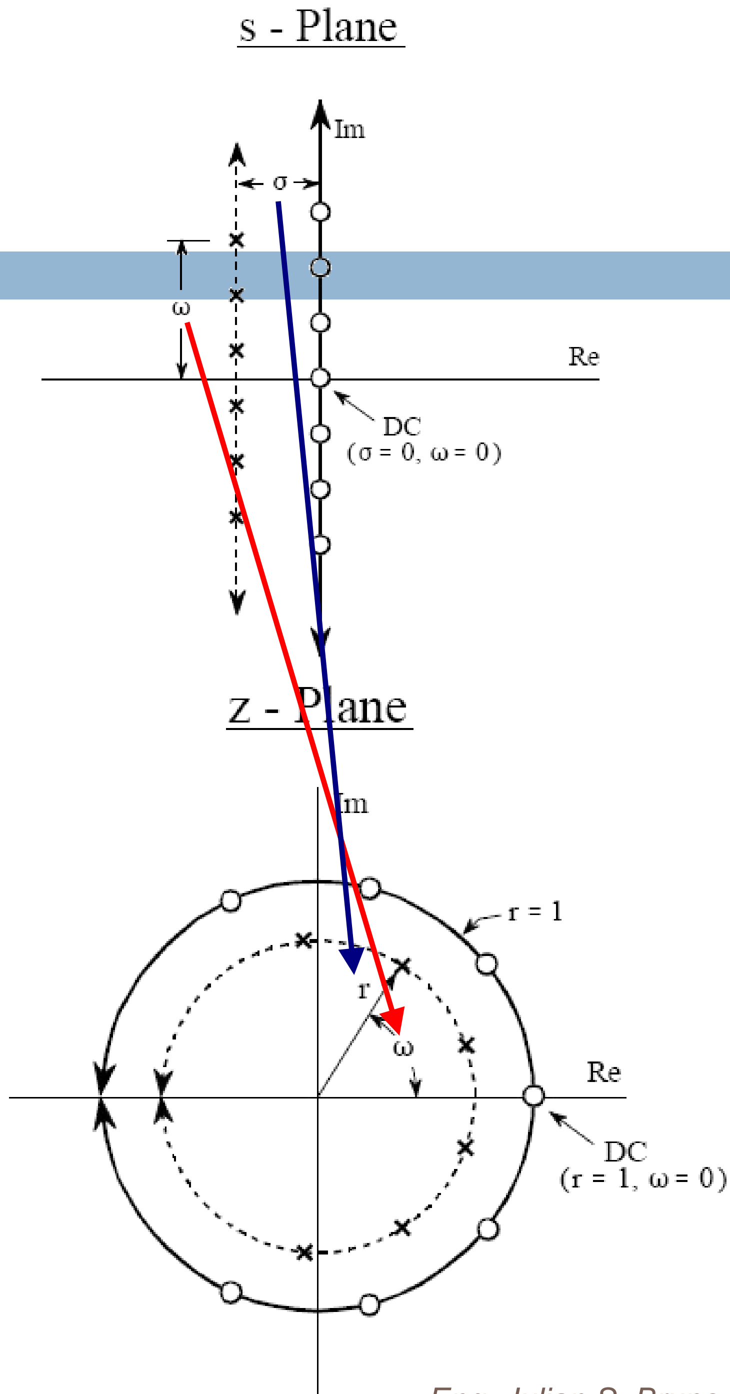
n samples delay

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

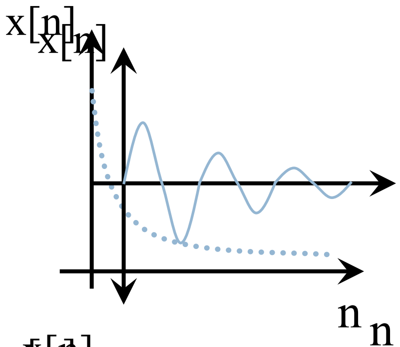
$$z = re^{j\omega}$$

Z Transform

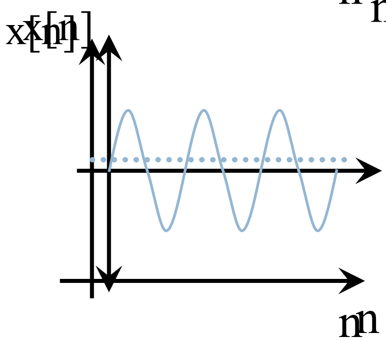
- Z transform is the discrete counterpart of Laplace transform.
- A vector in the Z plane have a Frequency equal to its argument and a damping equal to his modulo.
- It is used to show the behavior of digital systems.
- Similar to the continuous domain, DFT is a particular case of Z transform. **Which case?**



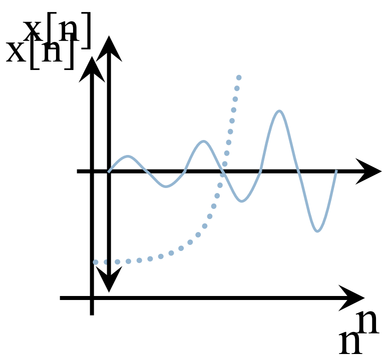
Z plane important places



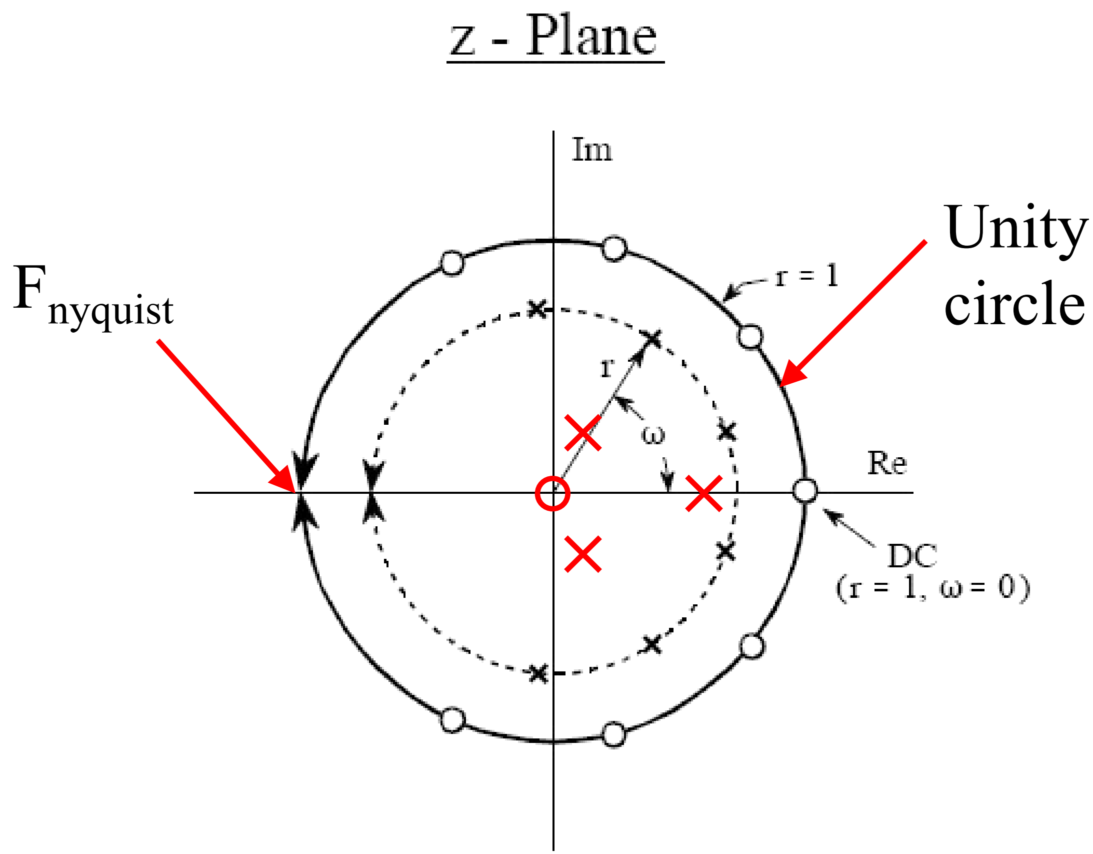
Stable



Marginally stable

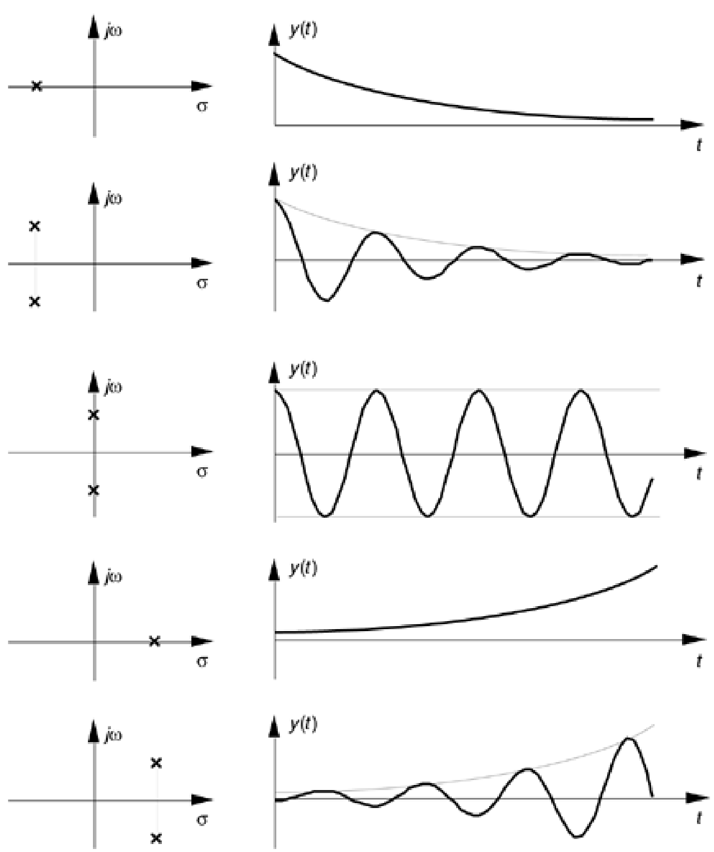


Unstable

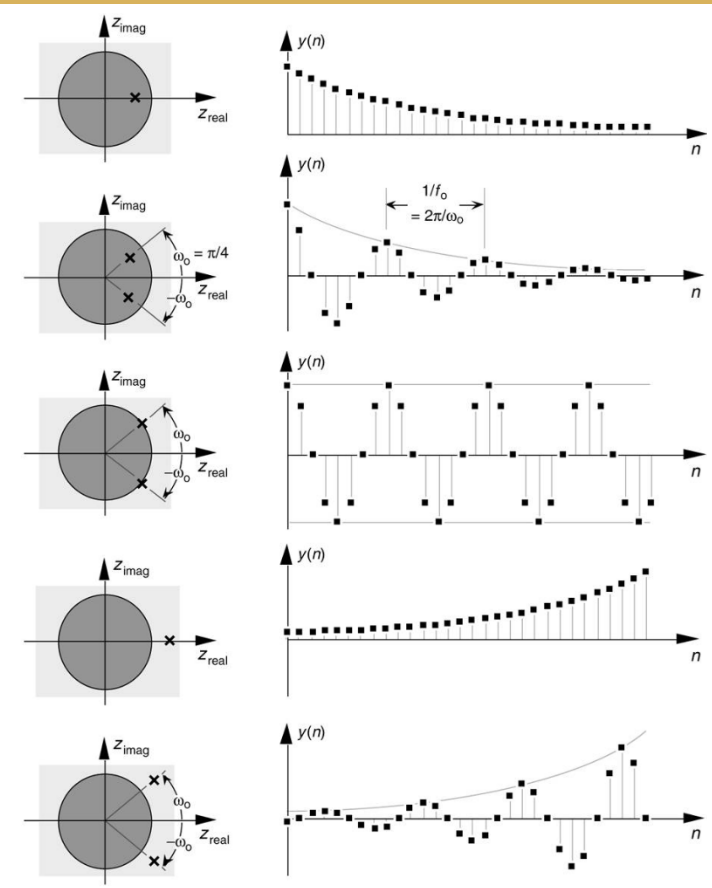


Pole locations and Time-domain impulse responses

S-plane



Z-plane



Particular Case – Fourier Transform

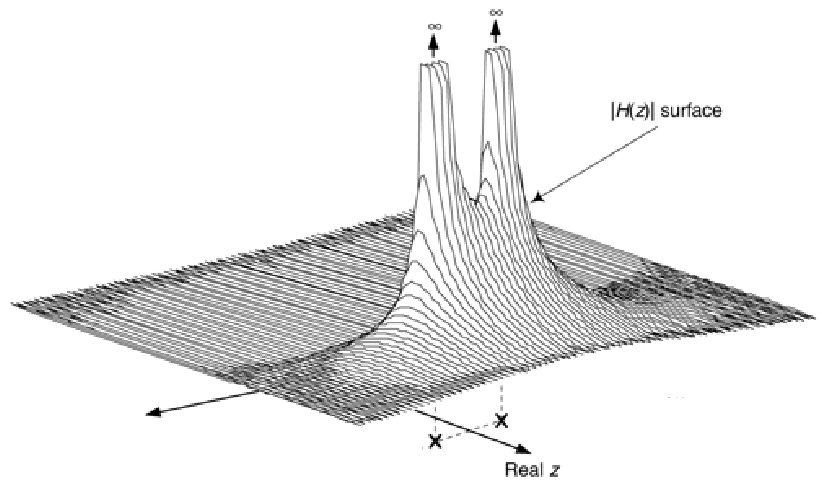
$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} \quad ; \quad z = re^{j\omega}$$

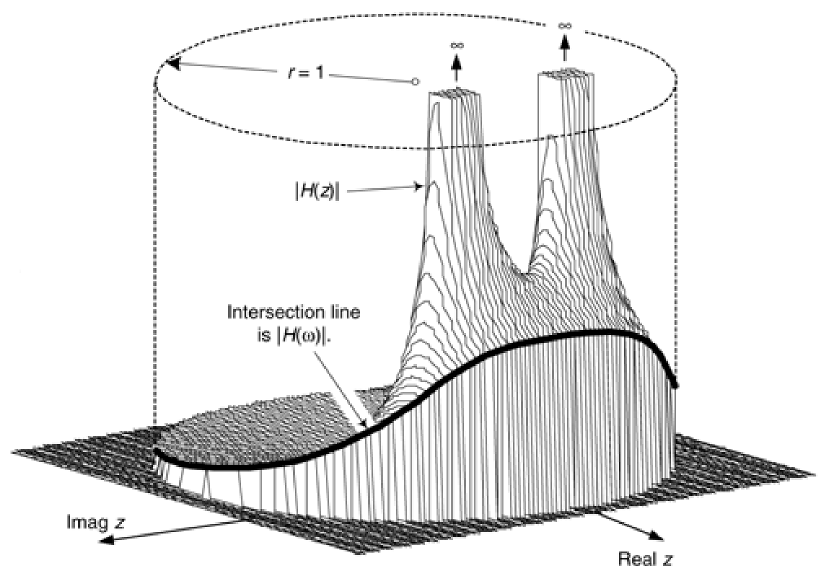
$$H(re^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] r^{-n} (e^{-j\omega n})$$

$$when \quad |z| = 1 \, , \, z = e^{j\omega}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

DTFT





Z Plane System representation

Difference equation

$$y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] + \cdots + b_1 y[n-1] + b_2 y[n-2] + b_3 y[n-3] + \cdots$$

Transfer Function

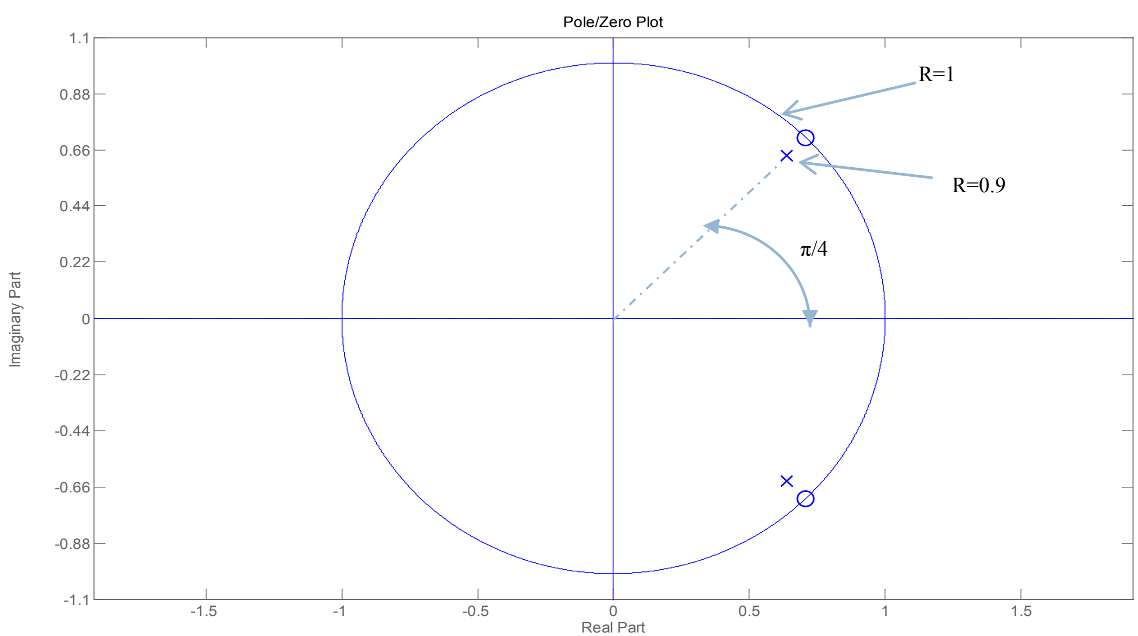
$$H[z] = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \cdots}{1 - b_1 z^{-1} - b_2 z^{-2} - b_3 z^{-3} - \cdots}$$

$$H[z] = \frac{(z - z_1)(z - z_2)(z - z_3)\cdots}{(z - p_1)(z - p_2)(z - p_3)\cdots}$$

- Difference equation form is useful to **implement** the system (i.e. DSP, Matlab, etc.)
- Transfer function is useful to **design and analyze** system's behavior by zero/pole placement/location.

Example: A notch filter

- A notch filter is a system that rejects only a particular frequency.
- This is equivalent to place a zero in this particular frequency, and a pole very close to the zero.



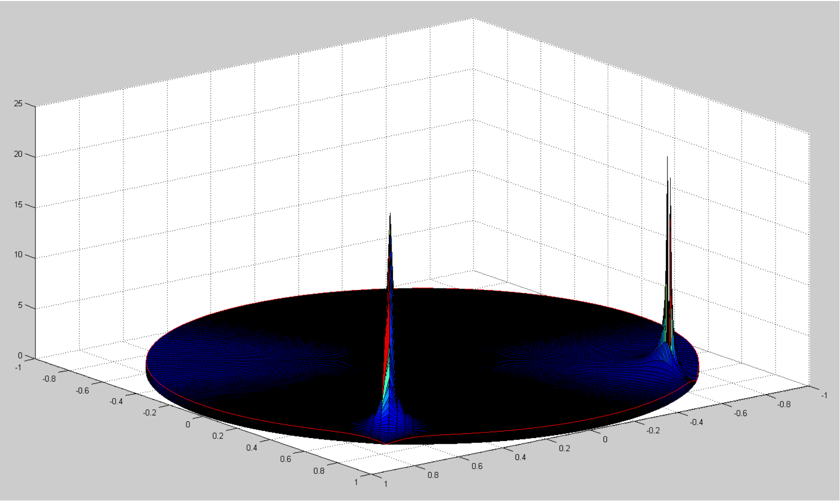
Example: A notch filter (II)

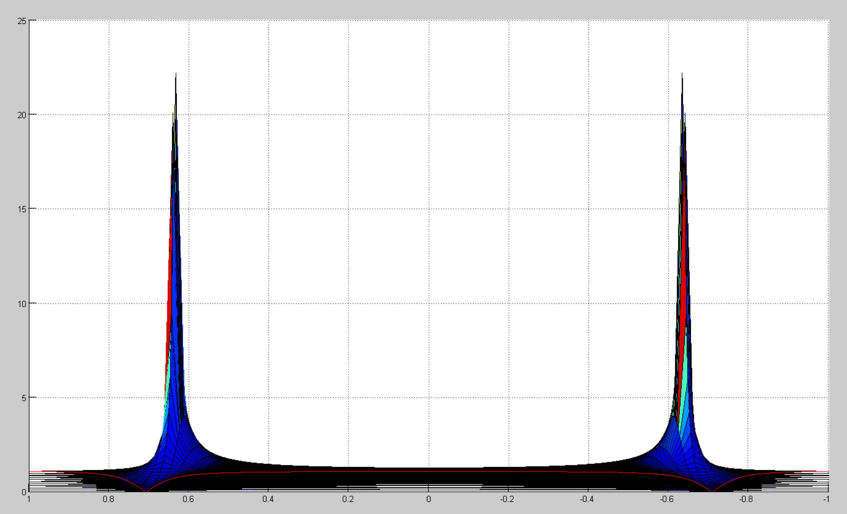
$$H(z) = \frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)}$$

Polar form	Rectangular form
$z_1 = 1e^{j\pi/4}$	$z_1 = 0.7071 + j0.7071$
$z_2 = 1e^{-j\pi/4}$	$z_2 = 0.7071 - j0.7071$
$p_1 = 0.9e^{j\pi/4}$	$p_1 = 0.6364 + j0.6364$
$p_2 = 0.9e^{-j\pi/4}$	$p_2 = 0.6364 - j0.6364$

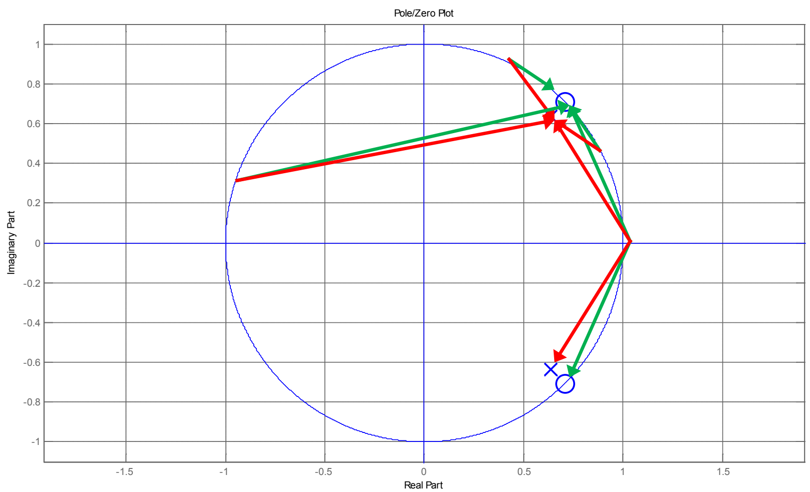
$$H(z) = \frac{1.000 - 1.414z + 1.000z^2}{0.810 - 1.273z + 1.000z^2}$$

$$H(z) = \frac{1.000 - 1.414z^{-1} + 1.000z^{-2}}{1.000 - 1.273z^{-1} + 0.810z^{-2}}$$



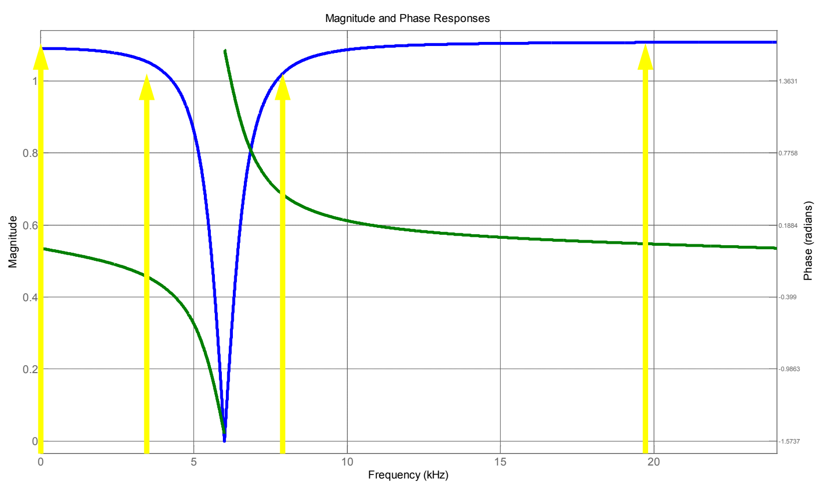


Frequency response of a notch filter



Modulo:

$$\left|H[z]\right| = \frac{\prod \left|v_{zero}\right|}{\prod \left|v_{pole}\right|}$$



Phase:

$$Arg\left(H[z]\right) = Arg\left(v_{zero}\right) - Arg\left(v_{pole}\right)$$

Implementation of a notch filter

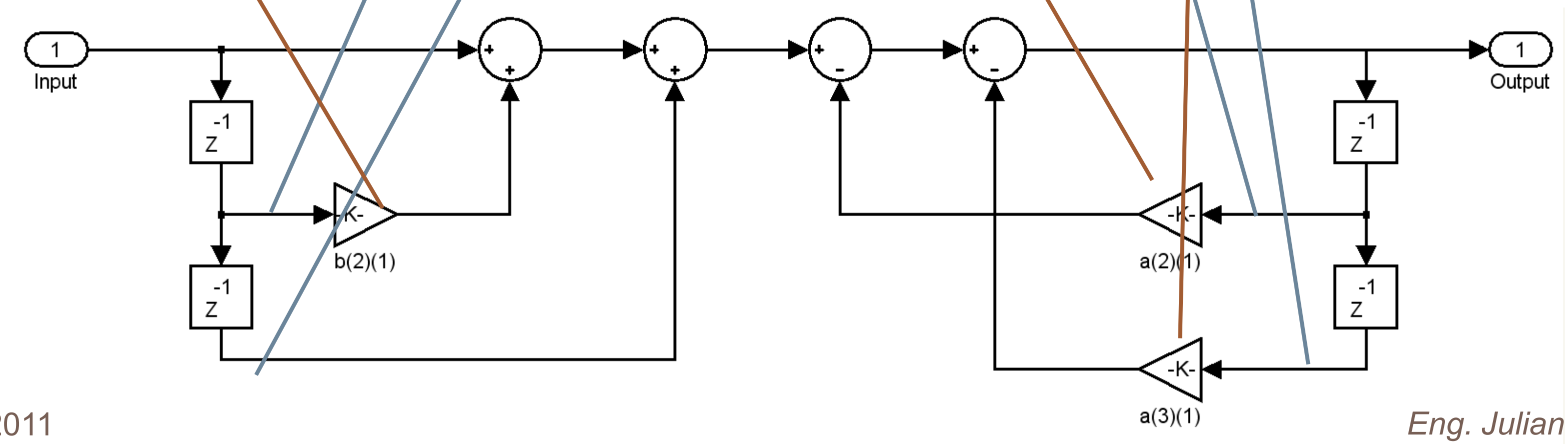
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1.000 - 1.414z^{-1} + 1.000z^{-2}}{1.000 - 1.273z^{-1} + 0.810z^{-2}}$$

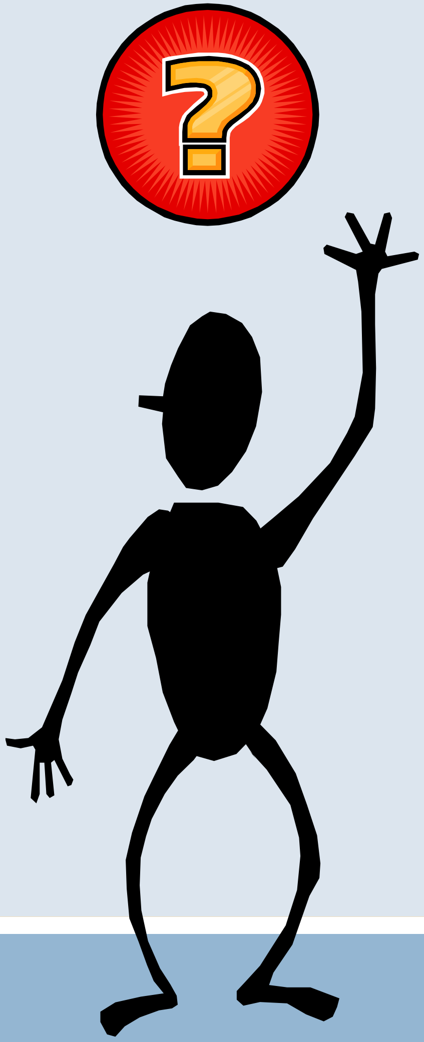
$$Y(z) = X(z)(1.000 - 1.414z^{-1} + 1.000z^{-2}) - Y(z)(-1.273z^{-1} + 0.810z^{-2})$$

$b(1) = 1.000 \qquad a(1) = 1.000$

$b(2) = -1.414 \qquad a(2) = -1.273$

$b(3) = 1.000 \qquad a(3) = 0.810$





Questions?

Thank you!